The Lyndon-Hochschild-Serre Spectral Sequence

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1 Introduction

In this section the notion of classifying spaces and group cohomology will be reviewed. These theories are necessary to get into the Lyndon-Hoschild-Serre Spectral sequence and study its applications.

Throughout this section G will denote a topological group.

Definition 1. A left G-space is a topological space X equipped with a continuous left G-action $G \times X \to X$. If X and Y are G-spaces, a G-equivariant map is a map $f : X \to Y$ such that $f(g \cdot x) = g \cdot f(x)$ for any $x \in X$, $g \in G$. A G-homotopy between G-maps f, g is a homotopy $F : X \times I \to Y$ in the usual sense, with the added condition that F be G- equivariant (here G acts trivially on the I coordinate).

This yields the G-homotopy category of left G-spaces. Similar definitions apply to right G-spaces.

Definition 2. let B be a topological space. Suppose that P is a right G-space equipped with a G-map $\pi : E \to B$ where G acts trivially on B We say that (P, π) principal G-bundle over B if it satisfies the following local triviality condition:

B has a covering by open sets U such that there exist G-equivariant homeomorphisms $\phi_U : \pi^{-1}(U) \to U \times G$ commuting in the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p_1} & \\ U & & \end{array}$$

Where $U \times G$ has the right G-action $(u, g)h = (u, gh)$.

Note this condition implies that G acts freely on P, and that π factors through a homeomorphism $P/G \to B$ (thus B "is" the orbit space of P).

Summarizing: A principal G-bundle over B consists of a locally trivial free G-space with orbit space B.

Given a principal G-bundle P over B and a map $f : B' \to B$, we can form the pullback $f^*P = B' \times_B P$ which inherits a natural structure of principal G-bundle over B'. The next results classify the principal G-bundles over a given topological space.

Lemma 3. *Let B be topological space and P a principal G-bundle over B, suppose that X is a CW-complex and that $f, g : X \to B$ are homotopic maps. Then the pullbacks f^*P and g^*P are isomorphic as principal G-bunldes over X.*

This result leads to the following classification theorem.

Theorem 4. *Suppose that there exists $P \to B$ a principal G-bundle with P contractible. Denote by $[X, B]$ the homotopy classes of maps $X \to B$ and $\mathcal{P}_G(X)$ the set of principal G-bundles over X up to isomorphism. Then the map $[X, B] \to \mathcal{P}_G(X)$ given by $f \mapsto f^*P$ is a bijection.*
Moreover, we have that

(i) B can be taken to be a CW-complex.

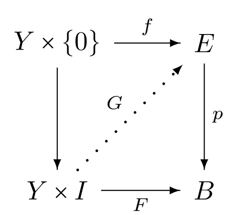
(ii) B is unique up to canonical homotopy equivalence.

(iii) P is unique up to G-homotopy equivalence.

We then call B a classyfing space for G and P a universal G-bundle. For this theory to be of any use, we need to know that classyfing spaces exist. The next theorem i s due to [Milnor]

Theorem 5. *Let G be a topological group group. Then there exists a classifying space BG and an universal G-bundle EG for G.*

Recall that a map $p : E \to B$ of topological spaces is called a fibration if for any topological space Y and any commutative diagram



That means, any homotopy $F : Y \times I \to B$ with initial condition $f : Y \to E$ can be lifted into a homotopy $G : Y \times I \to E$. In particular, if B is path connected, the space $F_b = p^{-1}(b)$ does not depend on the choice of b up to homotopy; we call $F := F_b$ the fiber.

In particular, if (E, π) is a principal G-bundle, it is a fibration with fiber G.

The main relation that we concern between principal G-bundles and fibrations is given by the following result.

Proposition 6. 1. *Suppose that H is an admissible subgroup of G, i.e. H is such that the induced action of H on G makes the map $G \to G/H$ a principal H-bundle, then there is a fibration induced by the inclusion $H \to G$*

$$BH \to BG$$

with fiber G/H up to weak equivalence. Moreover, the inclusion of the fiber $G/H \to BH$ classifies the principal H-bundle $G \to G/H$.

2. *Suppose that H is an admissible normal subgroup of G. Then there is a fibration $BG \to B(G/H)$ induced by the quotient map $G \to G/H$ with fiber BH up to weak equivalence.*

Definition 7. Let G be a topological group and BG its classifying space, we define the group cohomology ring of G with coefficients in the group A as the usual singular cohomology for topological spaces

$$H^*(G;A) := H^*(BG;A)$$

Now we pass to review some generalities on Homological algebra and the approach to group cohomology.

Let A, B, C, M be left R-modules. Recall that if the sequence

$$0 \to A \to B \to C \to 0$$

then

$$0 \to \hom(C, M) \to \hom(B, M) \to \hom(A, M)$$

is exact.

Now consider a projective resolution $P_* = \cdots P_1 \to P_0 \to A \to 0$ of R-modules over A. And consider the exact sequence

$$0 \to \hom(A, M) \to \hom(P_0, M) \to \hom(P_1, M) \to \cdots$$

And define

$$\mathrm{Ext}^n_R(A, M) := H_n(\hom(P_*, M))$$

that is, the n-th homology of the chain complex $\hom(P_*, M)$

Let G be a group and let $\mathbb{Z}G$ be its integral group ring. This means that as an additive group $\mathbb{Z}G$ is the free abelian group with the elements of G as a basis, and multiplication within the ring is determined by multiplication of the basis elements, which is multiplication in G. A typical element of $\mathbb{Z}G$ is a formal sum $\sum_{x \in G} \lambda_x x$ with $\lambda_x \in \mathbb{Z}$ where all but finitely many λ_x are zero.

The formula for multiplication of two general elements is

$$\sum_{x \in X} \lambda_x x \cdot \sum_{y \in G} \mu_y y = \lambda_{x,y \in G}(\lambda_x \mu_y) xy$$

We denote by $\mathbb{Z}$ the $\mathbb{Z}G$-module which is $\mathbb{Z}$ as an additive group, and where the action of G is trivial, i.e. $gn = n$ for all $n \in \mathbb{Z}$ and $g \in G$. This defines the (left) trivial module, the right trivial module being defined similarly.

Definition 8. Let G be a topological group, we define the n-th cohomology group of G with coefficients in the left $\mathbb{Z}G$-module M to be

$$H^n(G, M) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

The next result relate the two definitions of group cohomology given so far:

Theorem 9. *Let G be a topological group and R be a commutative ring, there is a natural map*

$$H^n(G, R) \otimes H^m(G, R) \to H^{n+m}(G, R)$$

that makes $H^*(G, R)$ into a graded commutative ring. Moreover, there is a ring isomorphism

$$H^*(G, R) \cong H^*(BG, R)$$

We now start to explore these cohomology groups by identifying them in low degrees and by construction of some particular resolutions of $\mathbb{Z}$. We define a mapping $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ by the assignment $g \mapsto 1$ for every $g \in G$, extended by linearity to the whole of $\mathbb{Z}G$. Thus the effect of ϵ on a general element of $\mathbb{Z}G$ is

$$\epsilon(\sigma_{g\in G}\lambda_g g) = \sigma_{g\in G}\lambda_g$$

This is the augmentation map and it is a ring homomorphism, and also a homomorphism of $\mathbb{Z}G$-modules. We write $IG := \ker(\epsilon)$ and this 2-sided ideal is called the augmentation ideal of $\mathbb{Z}G$. Because ϵ is surjective we may always use it to start a projective $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and evidently $\mathbb{Z} \cong \mathbb{Z}G/IG$. If G is finite we will also consider the element $N = g \in G, g \in \mathbb{Z}G$ which is sometimes called the norm element.

If M is a $\mathbb{Z}G$-module we write $M^G := \{m \in Mgm = m$ for all $g \in G\}$for the fixed points of G on M and $M_G := M/ < gm - mm \in M, g \in G >$ for the fixed quotient or cofixedpoints of G on M.

Proposition 10. *Let M be a* $\mathbb{Z}$G*-module.*

1. *The set $\{g - 1|1 \neq g \in G\}$ is a $\mathbb{Z}-$basis for IG.*

2. $H^0(G, M) = \hom_{\mathbb{Z}\mathrm{G}}(\mathbb{Z}, M) \cong M^G$. *The fixed point set M^G coincides with the set of elements of M anihilated by IG.*

The main example of cohomology of finite groups is the following

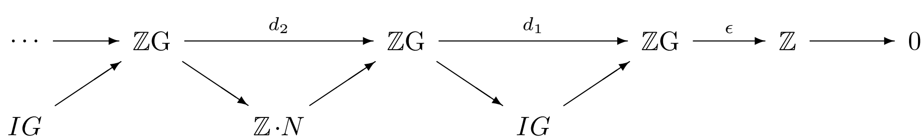
Example 11. Let $G = \langle g \rangle$ be a finite cyclic group and M a $\mathbb{Z}$G-module. Then for all $n \geq 1$ we have

$$H^{2n+1}(G, M) \cong H^1(G, M) \cong \ker(N)/(IG \cdot M)$$

and

$$H^{2n}(G, M) \cong H^2(G, M) \cong M^G/(N \cdot M)$$

To prove it, consider the following periodic resolution



where $d_1(1) = g - 1$ and $d_2(1) = N$. If we apply the functor $\hom_{\mathbb{Z}\mathrm{G}}(, M)$ we get a complex

$$0 \to M \xrightarrow{g-1} M \xrightarrow{N} M \xrightarrow{g-1} M \xrightarrow{N} M \to \cdots$$

where N and $g - 1$ denote the maps which are multiplication by these elements. We take the homology of that complex to obtain the result, using the fact that the kernel of $g - 1$ is the fixed point by Proposition 10.

2 Spectral Sequences

In this section we introduce the main computational technique for determine cohomology rings. We start with the general algebraic approach of a spectral sequence, that leads to particular geometrical cases such as the Serre spectral sequence for fibrations or the Lyndon-Hoschild-Serre spectral sequence for resolutions of $\mathbb{Z}$G-modules.

A $\mathbb{Z}$-bigraded module is a family $E = \{E_{p,q}\}$ one for each pair of indices $p, q = \pm 0, 1, 2, \ldots$. A differential $d : E \to E$ of bidegree $(-r, r-1)$ is a family of homomorphism $\{d : E_{p,q} \to E_{p-r,q+r-1}$ with $d^2 = 0$. The homology $H(E) = H(E, d)$ of E under this differential is the bigraded module $\{H_{p,q}(E)\}$ defined in the usual way as

$$H_{p,q}(E) = \ker(d : E_{p,q} \to E_{p-r,q+r-1})/dE_{p+r,q-r+1}$$

Definition 12. A spectral sequence $E = \{E^r, d^r\}$ is a sequence $E^2, E^3, \ldots$ of bigraded $\mathbb{Z}$-modules each with a differential d^r of bidegree $(-r, r-1)$ and with isomorphism

$$H(E^r, d^r) \cong E^{r+1}$$

for $r = 2, 3, \ldots$. If E' is another spectral sequence, a morphism of spectral sequences $f : E \to E'$ is a family of homomorphism $f^r : E^r \to E'^r$ of bigraded modules, each of bidegree $(0, 0)$ with $d^r f^r = f^r d^r$ and such that f^{r+1} is the map induced by f^r on homology.

A first quadrant spectral sequence E is one with $E^r_{p,q} = 0$ when $p < 0$ or $q < 0$. For cohomology, we write

$$E_r^{p,q} = E^r_{-p,-q}$$

and now the differentials are $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ of bedegree $(r, 1-r)$ and $H(E_r, d_r) \cong E_{r+1}$.

It is convenient to display the $E_r^{p,q}$ modules at the lattice points of the first quadrant of the p, q plane

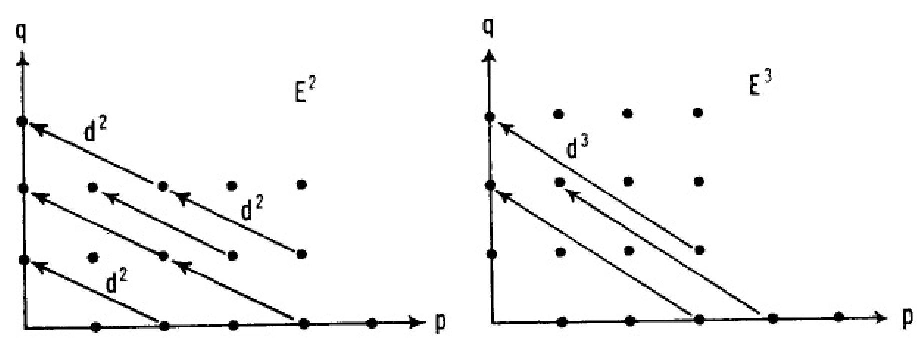


Figure 1: Page E_2 and E_3 of a homology spectral sequence

When the cohomology spectral sequence is considered, just reverse the direction of the arrow.

In general, for a homology spectral sequence, we have that $E^r \cong C^r/B^r$ where $C^r = \ker d^3$ and $B^r imd^r$. We have a tower

$$0 = B^1 \subseteq B^2 \subseteq \cdots \subseteq C^2 \subseteq C^1 = E_2$$

of bigraded submodules of E_2. Define the modules $C^\infty = \cap_{r\geq 2} C^r$ and $B^\infty = \cup_{r\geq 2} B^r$, then $B^\infty \subseteq C^\infty$, and therefore the spectral sequence determines a bigraded module

$$E^\infty_{p,q} \cong C^\infty_{p,q}/B^\infty_{p,q}$$

In this case, we say that the spectral sequence *converges* to E^∞, and we write

$$E^2 \Rightarrow E^\infty.$$

The following theorem is due to Serre [1951] following Leray's construction [1946,1950] that relates spectral sequences with fibrations.

Theorem 13 (Leray-Serre)**.** *Suppose that $f : E \to B$ is a fiber map with base B pathwise connected and simply connected, and fiber F pathwise connected. There is a first quadrant spectral sequence E_2 associated to the fibration such that*

$$E_2^{p,q} \cong H^p(B, H^q(F))$$

and

$$\bigoplus_{p+q=n} E_\infty^{p,q} \cong H^n(E)$$

Another way of constructing spectral sequences is via filtered modules. We begin with the following definitions

Definition 14. A filtration F of a module A is a family of submodules F_pA one for each $p \in \mathbb{Z}$ with

$$\cdots \subseteq F_{p-1}A \subseteq F_pA \subseteq F_{p+1}A \subseteq \cdots$$

Each filtration determines an associated graded modulde G^FA where

$$G^FA_p = F_pA \subseteq F_{p-1}A$$

. In the case where A is a differential $\mathbb{Z}$-graded module, the filtration induces a filtration on the $\mathbb{Z}$-graded module $H(A)$. Also, the family F_pA_n is a $\mathbb{Z}$ bigraded module.

A spectral sequence (E,d) is said to converge to a graded module H (in symbols, $E \to H$) if there is a filtration F of H and for each p isomorphisms $E_p^\infty \cong F_pH/F_{p-1}H$

The associated spectral sequence of a filtration may now be defined and given by the following result.

Theorem 15. *Each filtration F of a differential graded $\mathbb{Z}$-module A determines a spectral sequence (E^r, d^r) together with natural isomorphism*

$$E^1_{p,q} \cong H_{p+q}(F_pA/F_{p-1}A)$$

and

$$E^\infty_{p,q} \cong F_p(H_{p+q}A)/F_{p-1}H_{p+q}A$$

The filtration F of a DG-module A is canonically bounded if $F_{-1}A = 0$ and $F_nA_n, = A_n$, in each degree n.

Theorem 16. *If F is a canonically bounded filtration of a positively graded DG-module A , the spectral sequence of F lies in the first quadrant and the induced filtration of HA is finite, of the form*

$$0 = F_{n-1}H_nA \subseteq F_0H_nA \subseteq \cdots \subseteq F_nH_nA = H_nA$$

with successive quotients

$$F_pH_n/F_{p-1}H_n \cong E^\infty_{p,n-p}$$

under isomorphisms induced by 1_A. For example, the LERAY-SERRE theorem arises]tom a canonically bounded filtration of the singular chains o] a fiber space.

Many useful filtrations arise from bicomplexes. A bicomplex K is a family $K_{p,q}$ of modules with two families

$$\partial' K_{p,q} \to K_{p-1,q}$$

$$\partial'' K_{p,q} \to K_{p,q-1}$$

of module homomorphism, defined for all integers p, q and such that

$$\partial'^2 = \partial''^2 = 0$$

and

$$\partial'\partial'' + \partial''\partial' = 0$$

A bicomplex is positive if it lies in the first quadrant.

Each bicomplex K determines a single complexv $X = Tot(K)$ as

$$X_n = \sum_{p,q=n} K_{p,q}$$

with differential $\partial = \partial' + \partial'' : X_n \to X_{n-1}$ It is easy to check that $\partial^2 = 0$.

We have a filtration, called the first Filtration F' of X defined by the subcomplexes F'_p with

$$(F'_pX)_n = \sum_{k\leq p} K_{k,n-k}$$

The associated spectral sequence of F is called the *first spectral sequence* E' of the bicomplex.

Theorem 17. *For the first spectral sequence E' of a bicomplex K with associated total complex X there are natural isomorphism*

$$E'_{p,q}2 \cong H'_p H'' q(K)$$

If K is positive, E lies in the first quadrant and $E' \Rightarrow H(X)$.

We are in conditions to state and prove the main theorem for computing group cohomology

Theorem 18 (The Lyndon-Serre-Hoschild Spectral Sequence)**.** *Let N be a normal subgroup of G, and A be a $\mathbb{Z}G$-module. There is a first quadrant spectral sequence (E, d) natural in A, associated to the extension*

$$1 \to N \to G \to G/N$$

with natural isomorphism

$$E_2^{p,q} \cong H^p(G/N, H^q(N, A) \Rightarrow H^{p+q}(G, A)$$

3 Computations on Group Cohomology

In this section we apply the Leray-Serre spectral sequence, and the Lindon-Serre-Hoschild spectral sequence to compute the group cohomology of some groups.

Let $G = \mathbb{Z}/p\mathbb{Z}$ and $k = \mathbb{F}_p$. Assume that there is a finite length kG-resolution C_* of k where each C_n is a finitely generated permutation module. More precisely, since G has two subgroups, G and $\{1\}$, there exists finite sets I_n and J_n depending on n such that

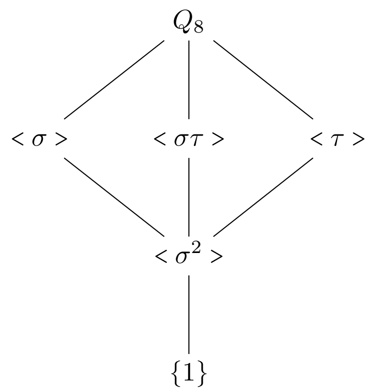
$$C_n = \oplus_{I_n} k[G/G] \oplus \oplus_{J_n} k[G/1].$$

We consider the subcomplex D_* such that $D_n = \oplus_{I_n} k[G/G] \cong \oplus_{I_n} k$. Now, let $i : D_* \to C_*$ be the inclusion of subcomplex. Let $P \to \mathbb{Z}$ be a $\mathbb{Z}G$-module projective resolution. We defined $H_*(G, C) := H_*(Tor(P \underset{\mathbb{Z}G}{\otimes} C))$ and $H_*(G, D) := H_*(Tor(P \underset{\mathbb{Z}G}{\otimes} D))$. Since G acts trivially on D_*, Künneth formula gives for every couple (p, q)

$$H_n(G, D) \cong (\oplus_{p+q=n} H_p(G, k) \otimes H_q D) \oplus (\oplus_{p+q=n-1} Tor_1(H_p(G, k), H_q D)).$$

Now k is a field so $Tor_1(H_p(G, k), H_q D) = 0$ for all p, q and $H_n(G, D) \cong \oplus_{p+q=n} H_p(G, k) \otimes_k H_q(D)$. $H_p(G, k) \cong k$, thus $H_n(G, D) \cong \oplus_{q=1}^n H_q(D)$.
Furthermore, we have a converging spectral sequence $E^2_{p,q} = H_p(G, H_q C) \Rightarrow H_{p+q}(G, C)$. But C is assumed to be of finite length so does the chain complex C/D and each term is a free kG-module so they are all H_*-acyclic. By proposition 5.6 page 170 of Brown's Cohomology of Groups, we have a converging spectral sequence $E^2_{p,q} = H_p(G, H_q(C/D)) \Rightarrow H_{p+q}((C/D)_G) \cong H_{p+q}(G, C/D)$.
But we have the following short exact sequence $0 \to D \to C \to C/D \to 0$ which induces a long exact sequence in cohomology. But for $i >> 0$ large enough we have $H_i((C/D)_G) = 0$ and so $H_j(G, C) \cong H_j(G, D)$ for all $j \geq i$. So $k = \oplus_{j \leq i} H_j(G, D)$.

Consider the following presentation of the quaternion group $Q_8 := \{\sigma, \tau | \sigma^4 = 1, \sigma^2 = \tau^2, \tau\sigma\tau^{-1} = \sigma^{-1}\}$. We know that the subgroups of Q_8 are



and that each of them is normal. More precisely, Q_8 is a group of order 8, each of the subgroups $< \sigma >, < \sigma\tau >$ and $< \tau >$ are cyclic maximal subgroups all isomorphic to the cyclic group $\mathbb{Z}/4\mathbb{Z}$, $< \sigma^2 >$ is the center of Q_8 and is isomorphic to the cyclic group $\mathbb{Z}/2\mathbb{Z}$ and $\{1\}$ is the trivial subgroup. Therefore, the derived subgroup is $[Q_8, Q_8] = \mathbb{Z}/2\mathbb{Z}$ and the abelianization of Q_8, $Q_8/[Q_8, Q_8]$, is the Klein group $< [\sigma], [\tau] > = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We therefore have the following central group extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Q_8 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1.$$

We know that

$$H^1(Q_8, \mathbb{Z}/2\mathbb{Z}) \cong Hom_{\text{Groups}}(Q_8, \mathbb{Z}/2\mathbb{Z}) \cong Hom_{\text{Ab}}(Q_8/[Q_8, Q_8], \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

That is, $H^1(Q_8, \mathbb{Z}/2\mathbb{Z})$ is the dual space to the $\mathbb{F}_2$-vector space $Q_8/[Q_8, Q_8]$ of basis $\{[\sigma], [\tau]\}$. Hence, the dual basis made of the two following maps $x, y : Q_8/[Q_8, Q_8] \to \mathbb{Z}/2\mathbb{Z}$ characterized by

$$x([\sigma]) = 1, x([\tau]) = 0, \; y([\sigma]) = 0 \text{ and } y([\tau]) = 1.$$

From the third assignment we know that $H^*(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[w]$ with $|w| = 1$ and as seen in class, using Künneth formula, we find $H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[x, y]$ with $|x| = |y| = 1$. Now, the quotient group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = Q_8/[Q_8, Q_8]$ acts trivially on $H^*([Q_8, Q_8], \mathbb{Z}/2\mathbb{Z})$ since $[Q_8, Q_8] = \mathbb{Z}/2\mathbb{Z}$ is the center of Q_8. Therefore, the E_2-page of the Lyndon-Hochschild-Serre spectral sequence of the central extension

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Q_8 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1$$

is

$$E^2_{*,*} = H^*\Big(Q_8/[Q_8, Q_8], H^*([Q_8, Q_8], \mathbb{Z}/2\mathbb{Z})\Big) \cong H^*(Q_8/[Q_8, Q_8], \mathbb{Z}/2\mathbb{Z}) \otimes H^*([Q_8, Q_8], \mathbb{Z}2) \cong \mathbb{F}_2[x, y, w]$$

where we used the universal coefficient theorem to get the first isomorphism. We want to compute the image of $d_2 : E_2^{0,1} \to E_2^{2,0}$. On one hand, $H^1(Q_8, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = E_2^{1,0} = E_\infty^{1,0}$ so $d_2(w) \neq 0$. On the other hand, $E_2^{2,0} = H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, H^0(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})) = H^2(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is generated by degree 2 homogeneous polynomials in the variables x and y. Thus, there exists $a, b, c \in \mathbb{F}_2$ such that $d_2(w) = ax^2 + bxy + cy^2$.

To find the coefficients, we use the restriction map on the subgroups of Q_8. We first consider the following map of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Q_8 & \longrightarrow & <[\sigma],[\tau]> & \longrightarrow & 1 \\ & & \uparrow \cong & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & <\sigma^2> & \longrightarrow & <\sigma> & \longrightarrow & <[\sigma]> & \longrightarrow & 1 \end{array}$$

We define the restriction map $res_\sigma := res_{<[\sigma],[\tau]>.<[\sigma]>} : H^1(<[\sigma],[\tau]>, \mathbb{Z}/2\mathbb{Z}) \to H^1(<[\sigma]>, \mathbb{Z}/2\mathbb{Z})$ by $x \mapsto res_\sigma(x)$ a generator of $H^1(<[\sigma]>, \mathbb{Z}/2\mathbb{Z})$ and $y \mapsto 0$.

We now look at the Lyndon-Hochschild-Serre spectral sequence associated to the group extension

$$1 \to <\sigma^2> \to <\sigma> \to <[\sigma]> \to 1$$

(where we recall that $<\sigma^2> = \mathbb{Z}/2\mathbb{Z}, <\sigma> = \mathbb{Z}/4\mathbb{Z}$ and $<[\sigma]> = \mathbb{Z}/2\mathbb{Z}$). As before, the group $<[\sigma]>$ acts trivially on $H^*(<\sigma>, \mathbb{Z}/2\mathbb{Z})$ so by the universal coefficient theorem the ${}^\sigma E_2$-page is $H^*(<\sigma^2>, \mathbb{Z}/2\mathbb{Z}) \otimes H^*(<[\sigma]>, \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[x_1, x_2]$ with $|x_1| = |x_2| = 1$. Since $H^1(<\sigma>, \mathbb{Z}/2\mathbb{Z}) = Hom(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ the differential $\sigma_2^d :^\sigma E_2^{0,1} \to^\sigma E_2^{2,0}$ is nonzero because otherwise we would have $H^1(\mathbb{Z}/4, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ which can't be because ${}^\sigma E_2 = H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ converges to $H^*(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$. Now, using the naturality of spectral sequences we get the following commutative diagram

$$\begin{array}{ccc} E_2^{0,1} & \xrightarrow{d_2} & E_2^{2,0} \\ \cong \downarrow & & \downarrow res_\sigma \\ {}^\sigma E_2^{0,1} & \xrightarrow{{}^\sigma d_2} & {}^\sigma E_2^{2,0}. \end{array}$$

Hence by commutativity, $0 \neq res_\sigma(d_2(w)) = res_\sigma(ax^2 + bxy + cy^2) = a res_\sigma(x)^2$. But $res_\sigma(x)$ generates $H^1(<[\sigma]>, \mathbb{Z}/2\mathbb{Z})$ so $a = 1$.

We now consider the following map of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Q_8 & \longrightarrow & <[\sigma],[\tau]> & \longrightarrow & 1 \\ & & \uparrow \cong & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & <\sigma^2> & \longrightarrow & <\tau> & \longrightarrow & <[\tau]> & \longrightarrow & 1 \end{array}$$

By definition, $<[\tau]> = <[\sigma]>$

We define the restriction map $res_\tau := res_{<[\sigma],[\tau]>.<[\tau]>} : H^1(<[\sigma],[\tau]>, \mathbb{Z}/2\mathbb{Z}) \to H^1(<[\tau]>, \mathbb{Z}/2\mathbb{Z})$ by $x \mapsto 0$ and $y \mapsto res_\tau(y)$ a generator of $H^1(<[\tau]>, \mathbb{Z}/2\mathbb{Z})$.

We now look at the Lyndon-Hochschild-Serre spectral sequence associated to the group extension

$$1 \to< \sigma^2 >\to< \tau >\to< [\tau] >\to 1$$

As before, the group $< [\tau] >$ acts trivially on $H^*(< \tau >, \mathbb{Z}/2\mathbb{Z})$ so by the universal coefficient theorem the ${}^\tau E_2$-page is $H^*(< \sigma^2 >, \mathbb{Z}/2\mathbb{Z}) \otimes H^*(< [\tau] >, \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[x_3, x_4]$ with $|x_3| = |x_3| = 1$. Since $H^1(< \tau >, \mathbb{Z}/2\mathbb{Z}) = Hom(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ the differential $\tau_2^d :^\tau E_2^{0,1} \to^\tau E_2^{2,0}$ is nonzero for the same reason as before. Now, using the naturality of spectral sequences we get the following commutative diagram

$$\begin{array}{ccc} E_2^{0,1} & \xrightarrow{d_2} & E_2^{2,0} \\ \cong\downarrow & & \downarrow res_\tau \\ {}^\tau E_2^{0,1} & \xrightarrow{{}^\tau d_2} & {}^\tau E_2^{2,0}. \end{array}$$

Hence by commutativity, $0 \neq res_\tau(d_2(w)) = res_\tau(ax^2 + bxy + cy^2) = cres_\tau(y)^2$. But $res_\tau(y)$ generates $H^1(< [\tau] >, \mathbb{Z}/2\mathbb{Z})$ so $c = 1$.

We finally consider the following map of group extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Q_8 & \longrightarrow & < [\sigma], [\tau] > & \longrightarrow & 1 \\ & & \cong\uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & < \sigma^2 > & \longrightarrow & < \sigma\tau > & \longrightarrow & < [\sigma\tau] > & \longrightarrow & 1 \end{array}$$

By definition, $< [\sigma\tau] >=< [\sigma] >$

We define the restriction map $res_\sigma\tau := res_{<[\sigma],[\tau]>.<[\sigma\tau]>} : H^1(< [\sigma], [\tau] >, \mathbb{Z}/2\mathbb{Z}) \to H^1(< [\sigma\tau] >, \mathbb{Z}/2\mathbb{Z})$ by $x^2 \mapsto 0, y^2 \mapsto 0$ and $xy \mapsto res_{\sigma\tau}(xy)$ a generator of $H^1(< [\sigma\tau] >, \mathbb{Z}/2\mathbb{Z})$.

We now look at the Lyndon-Hochschild-Serre spectral sequence associated to the group extension

$$1 \to< \sigma^2 >\to< \sigma\tau >\to< [\sigma\tau] >\to 1$$

As before, the group $< [\sigma\tau] >$ acts trivially on $H^*(< \tau >, \mathbb{Z}/2\mathbb{Z})$ so by the universal coefficient theorem the ${}^{\sigma\tau} E_2$-page is $H^*(< \sigma^2 >, \mathbb{Z}/2\mathbb{Z}) \otimes H^*(< [\sigma\tau] >, \mathbb{Z}/2\mathbb{Z}) = H^*(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{F}_2[x_5, x_6]$ with $|x_5| = |x_6| = 1$. Since $H^1(< \sigma\tau >, \mathbb{Z}/2\mathbb{Z}) = Hom(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ the differential $\sigma\tau_2^d :^{\sigma\tau} E_2^{0,1} \to^{\sigma\tau} E_2^{2,0}$ is nonzero for the same reason as before. Now, using the naturality of spectral sequences we get the following commutative diagram

$$\begin{array}{ccc} E_2^{0,1} & \xrightarrow{d_2} & E_2^{2,0} \\ \cong\downarrow & & \downarrow res_{\sigma\tau} \\ {}^{\sigma\tau} E_2^{0,1} & \xrightarrow{{}^{\sigma\tau} d_2} & {}^{\sigma\tau} E_2^{2,0}. \end{array}$$

Hence by commutativity, $0 \neq res_{\sigma\tau}(d_2(w)) = res_{\sigma\tau}(ax^2 + bxy + cy^2) = bres_{\sigma\tau}(xy)$. But $res_{\sigma\tau}(xy)$ generates $H^1(<[\sigma\tau>, \mathbb{Z}/2\mathbb{Z})$ so $b = 1$.
This shows that $d_2(w) = x^2 + xy + y^2$.

Now we compute the cohomology of the semi-direct product group $G = (S^1)^r \rtimes \mathbb{Z}/2$ with *mod* 2 coefficients for $r \geq 1$. Here $H^*(\cdot)$ denotes $H^*(\cdot, \mathbb{Z}/2)$

Let T denote $(S^1)^r$. Recall that $H^*(BT) \cong \mathbb{Z}/2[c_1, \ldots, c_r]$, $|c_i| = 2$ for all $i = 1, \ldots, r$ and $H^*(B\mathbb{Z}/2) \cong \mathbb{Z}/2[w]$, $|w| = 1$ (See [**?**, Th 14.5].) To compute the cohomology of G, the short exact sequence

$$1 \to T \to G \to \mathbb{Z}/2 \to 1$$

yields in a fibration of classifying spaces

$$BT \to BG \to B\mathbb{Z}/2$$

where the E_2 page of the associated Leray-Serre Spectral sequence is given by

$$E_2^{p,q} \cong H^p(B\mathbb{Z}/2; H^q(BT)) \Rightarrow H^*(BG)$$

since $\mathbb{Z}/2$ acts on $H^q(BT)$ trivially. (In general the action of $\mathbb{Z}/2 = \{\pm 1\}$ over $H^q(BT;\mathbb{Z})$ is given by the induced action on the generators $\pm 1 \cdot c_i = \pm c_i$).

Therefore, by the universal coefficient theorem we have a $\mathbb{Z}/2$-algebras isomorphism

$$E_2 \cong H^*(B\mathbb{Z}/2) \otimes H^*(BT) \cong \mathbb{Z}/2[w, c_1, \ldots, c_r],$$

and thus, the differential d_2 depends only on the values on the generators w and c_i because it is a derivation. Namely, $d_2(w) = 0$ since w lies on the x-axis of the spectral sequence and $d_2(c_i) = 0$ for all $i = 1, \ldots, r$ since $d_2(c_i) \in E_2^{2,1} = H^2(B\mathbb{Z}/2) \otimes H^1(BT) = 0$.

It follows that $d_2 = 0$, implying that $E_3 \cong E_2$. Now we consider the differential d_3; as before, we only need to compute the map $d_3 : E_3^{0,2} \to E_3^{3,0}$. In this case, we have $d_3(c_i) = \alpha_i w^3$ with either $\alpha_i = 0$ or $\alpha_i = 1$.

The sub-extension

$$\begin{array}{ccccc} T & \longrightarrow & T \rtimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \\ \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & 1 \rtimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \end{array}$$

induces a map of spectral sequences $E_s^{p,q} \to \widetilde{E}_s^{p,q}$, where $\widetilde{E}_2 \cong H^*(B\mathbb{Z}/2)$ is the $\widetilde{E}_2$ page of the spectral sequence associated to the bottom exact sequence. By the naturarlity of the spectral sequences we have then a commutative diagram

$$\begin{array}{ccc} E_3^{0,2} & \xrightarrow{d_3} & E_3^{3,0} \\ \downarrow & & \downarrow \\ \widetilde{E}_3^{0,2} & \xrightarrow{\widetilde{d}_3} & \widetilde{E}_3^{3,0} \end{array}$$

which implies that $d_3 = 0$ since the right vertical arrow is the identity map and $\widetilde{d}_3 = 0$.

Notice that for $r \geq 4$ $E_r^{r,3-r} = 0$ an so is the differential $d_r : E_r^{0,2} \to E_r^{r,3-r}$. Therefore, the spectral sequence degenerates at page 2 and this implies that

$$E_2 \cong E_\infty \cong H^*(B\mathbb{Z}/2) \otimes H^*(BT) \cong \mathbb{Z}/2[w, c_1, \ldots, c_r] \cong H^*(BG)$$

Recall that the above isomorphism is a graded $\mathbb{Z}/2[w]$-module isomorphism; however since $H^*(BT)$ is a polynomial algebra, we can choose a multiplicative section $\varphi : H^*(BT) \to H^*(BG)$ of the surjective map $H^*(BG) \to H^*(BT)$. It follows from the Leray-Hirsch Theorem that such map together with the canonical map $p^* : H^*(B\mathbb{Z}/2) \to H^*(BG)$ give rise to an isomorphism of graded $H^*(B\mathbb{Z}/2)$-modules

$$\theta : H^*(B\mathbb{Z}/2) \otimes H^*(BT) \to H^*(BG)$$

given by $\theta(\alpha \otimes \beta) = \varphi(a)p^*(\beta)$. Moreover, the map θ is an isomorphism of graded $\mathbb{Z}/2$-algebras since both φ and p^* are multiplicative maps.

Furthermore, the restriction maps $H^*(BG) \to H^*(B\mathbb{Z}/2)$ and $H^*(BG) \to H^*(BT)$ induced by the inclusions, coincide with the projection of $H^*(B\mathbb{Z}/2) \otimes H^*(BT)$ on each factor respectively via the isomorphism θ.

Summarizing, we have

Proposition 19. *There is a graded $\mathbb{Z}/2$-algebra isomorphism $H^*(BG) \cong \mathbb{Z}/2[w, c_1, \ldots, c_r]$ such that the canonical maps $H^*(BG) \to H^*(BT)$, $H^*(BG) \to H^*(B\mathbb{Z}/2)$ and $H^*(B\mathbb{Z}/2) \to H^*(BG)$ coincide with the canonical restriction maps $\mathbb{Z}/2[w, c_1, \ldots, c_n] \to \mathbb{Z}/2[c_1, \ldots, c_n]$, $\mathbb{Z}/2[w, c_1, \ldots, c_n] \to \mathbb{Z}/2[w]$, and the canonical inclusion map $\mathbb{Z}/2[w] \to \mathbb{Z}/2[w, c_1, \ldots, c_n]$ respectively.*

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