Equivariant Cohomology

for a torus action with a compatible involution.

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Motivation

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Let $T = (S^1)^n$ be a torus acting on a compact symplectic manifold (M, ω) in a Hamiltonian way. Let

$$M^T = \{x \in M : g \cdot x = x, \ \forall g \in T\}$$

denote the fixed point subspace.

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denote the fixed point subspace. Then

Theorem (T. Frankel. 1959)

$$H^*(M; k) \cong \bigoplus_{i=1}^{m} H^{*-d_i}(F_i; k),$$

where $F_1, \ldots, F_m$ are the connected components of M^T, d_i is the Morse-Bott index associated to F_i and $\mathrm{char}(k) = 0$.

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How about the cohomology of the orbit space $H^*(X/G)$?

Example

Let $X = S^2$ and $G = \mathbb{Z}/2$ be the antipodal action on X. Then $H^*(X/G) \cong H^*(\mathbb{R}P^2)$.

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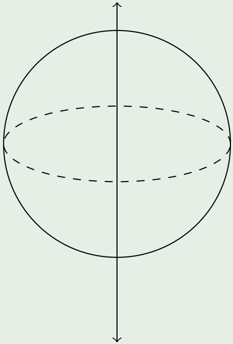

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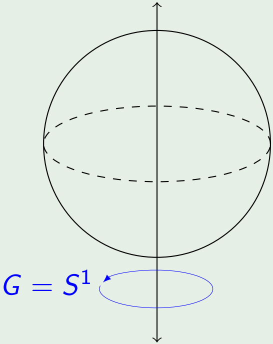

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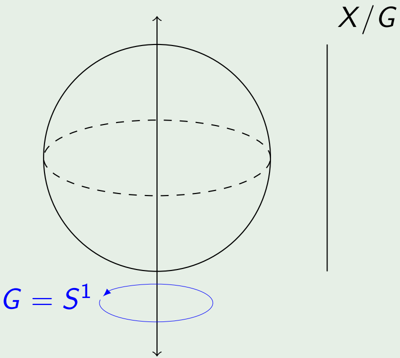

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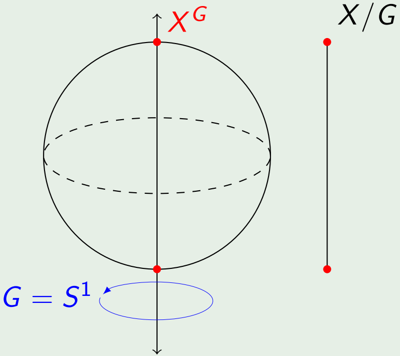

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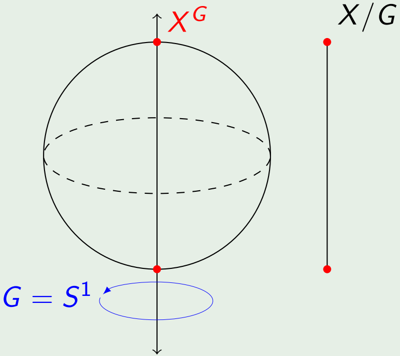

Here, $H^*(X/G) \cong H^*(\{pt\})$.

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Later we will see that $H^*(\tilde{S}^2/S^1)$ is non-trivial.

Borel's amazing idea

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The orbit space $BG := EG/G$ is called *the classifying space of G*.

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Example

- $G = S^1$, $EG = S^\infty$, $BG = \mathbb{C}P^\infty$.
- $G = \mathbb{Z}/2$, $EG = S^\infty$, $BG = \mathbb{R}P^\infty$.
- $G = \mathbb{Z}$, $EG = \mathbb{R}$, $BG = S^1$.

Equivariant Cohomology

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Definition (Seminar on transformation groups - A. Borel. 1960.)

For a G-space X, the Borel construction of X is the space $X_G = (EG \times X)/G$ and the G-equivariant cohomology of X is defined as

$$H^*_G(X) := H^*(X_G).$$

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$$H^*_G(X) := H^*(X_G).$$

For any G-spaces X, Y and a G-equivariant map $f : X \to Y$ (i.e. $f(g \cdot x) = g \cdot f(x)$), there is an induced map

$$f^*_G : H^*_G(Y) \to H^*_G(X).$$

Particular group actions

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Then for any G-space X, $H_G^*(X^G) \cong H^*(BG) \otimes H^*(X^G)$.

- If G acts on X freely, we have

$$X_G \simeq X/G \text{ and } H_G^*(X) \cong H^*(X/G).$$

A module structure on $H^*_G(X)$

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In particular, X^G is G-equivariantly formal.

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For $z \in EG$, the section $i_z : X \to X_G$ given by $i_z(x) = [z, x]$ induces a map

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Proposition

Let X be a G-space. The following are equivalent:

- *X is G-equivariantly formal.*
- *The map $r : H^*_G(X) \to H^*(X)$ is surjective.*
- *$H^*_G(X)$ is a free $H^*(BG)$-module (if G is connected).*

Reduction to torus actions

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So we can restrict the study of equivariant formality of compact Lie group actions to torus actions.

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Under this theorem, S^2 with the rotation action of S^1 is S^1-equivariantly formal. That is,

$$H^*_{S^1}(S^2) \cong H^*(BS^1) \otimes H^*(S^2).$$

Equivariant formality on Symplectic manifolds

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The Frankel-Atiyah's theorem

$$H^*(M;k) \cong \bigoplus_{i=1}^{m} H^{*-d_i}(F_i;k), \quad \left(\bigcup_{i=1}^{m} F_i = M^T \right)$$

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implies that $b(M) = b(M^T)$. So we get

Theorem

For a Hamiltonian action of a torus T on a symplectic manifold M, M is T-equivariantly formal and

$$H_T^*(M) \cong H^*(BT) \otimes H^*(M) \cong \bigoplus_{i=1}^{m} H_T^{*-d_i}(F_i).$$

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An action of $\mathbb{Z}/2$ on a topological space X is equivalent to an involution $\tau : X \to X$.

Reduction to 2-torus actions

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Let $T = (S^1)^n$ be a torus and consider the 2-torus subgroup of $T_2 = \{g \in T : g^2 = e\} \cong (\mathbb{Z}/2)^n$.

Theorem (M. Franz - S.)

For X a T-space, X is T-equivariantly formal if and only if X is T_2-equivariantly formal.

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Equivariant formality on $\text{char}(k) = 2$ reduces to study involutions.

Mixing torus action and involutions

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Let T be a torus, X be a T-space and $\tau : X \to X$ an involution. We say that τ *is compatible with the action of* T if for any $g \in G$, $x \in X$.

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Definition

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Definition

The fixed point subspace X^τ is called the *real locus* of X. X^τ inherits a natural action of the 2-torus $T_2 \subseteq T$.

Anti-symplectic involutions

Let (M, ω) be a symplectic manifold with a Hamiltonian action of a torus T. An anti-symplectic involution $\tau : M \to M$ is a smooth involution such that $\tau^*\omega = -\omega$.

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Theorem (H. Duistermaat. 1983)

Let M be a symplectic manifold with a Hamiltonian action of a torus T and a compatible anti-symplectic involution τ. Then

$$H^*(M^\tau) \cong \bigoplus_{i=1}^{m} H^{*-\frac{d_i}{2}}(F_i^\tau)$$

and $b(M^\tau) = b(M^\tau \cap M^T)$*, where* $M^T = \bigcup_{i=1}^{m} F_i$.

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Theorem (D. Biss - V. Guillemin - T. Holm - 2004)

- $$H^*_{T_2}(M^\tau) \cong \bigoplus_{i=1}^{m} H^{*-\frac{d_i}{2}}_{T_2}(F_i^\tau).$$
- $b(M^\tau) = b(M^\tau \cap M^{T_2}) = b((M^\tau)^{T_2})$ *and thus*
$$H^*_{T_2}(M^\tau) \cong H^*(BT_2) \otimes H^*(M^\tau).$$

Symplectic case: Moral of the story

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Does this situation hold in a more general setting?

The topological case

For just topological spaces, the answer is no!.

Example

Let $X = S^3 \subseteq \mathbb{C}^2$, let $T = S^1$ act on X by $g \cdot (u, z) = (gu, z)$ and let τ be the involution $\tau(u, z) = (\bar{u}, -z)$.

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$$b(X) = b(X^T) = b(X^\tau) = 2 \text{ and } b((X^\tau)^{T_2}) = 0.$$

Therefore, X is T-equivariantly formal but its real locus X^τ is not T_2-equivariantly formal.

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Theorem (M. Franz - S.)

Assume $\operatorname{char}(k) = 2$. *If X is G-equivariantly formal then its real locus X^τ is T_2-equivariantly formal.*

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Assume $\mathrm{char}(k) = 2$. *If X is G-equivariantly formal then its real locus X^τ is T_2-equivariantly formal.*

THANKS!!!

References

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