Clifford Algebras and Milnor's Conjecture for $n = 2$. Milnor K-Theory Final Project

Sergio Chaves

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Abstract

In this document, a step-by-step proof of the Milnor's Conjecture for $n = 2$ is presented. Relating apparently different structures over a field F such that the *Milnor K-theory, the Brauer Group and the Witt Ring* was a big field of study during the last decades. In the first sections all the basic notions and key algebraic objects mentioned before are introduced; most result are without proofs which can be found in [Lam]. Then it is focused in studying the properties of Quaternion Algebras and Clifford algebras, since the core of the proof of Milnor's Conjecture relies intrinsically in these objects. Throughout the whole document, F will denote a field with $char(F) \neq 2$.

1 Algebras and Quadratic Forms

Definition 1.1. A F**-algebra** is a F-vector space A, togheter with a bilinear operation $\cdot : A \times A \to A$ called *multiplication*, which makes A into a ring with unity 1. Given two F-algebras A, B, a F-algebra homomorphism $h : A \to B$ is a linear map that is also a ring homomorphism.

For instance, the ring $M_n(F)$, of all $n \times n$ matrices over F is a F-algebra.

And **ideal** $I \subseteq A$ is a linear subspace of A that is also a two-sided ideal with respect to multiplication in A.

Definition 1.2. Given two F-vector spaces V, W, a **tensor product** of V and W is a pair $(V \otimes W, \varphi)$ where $V \otimes W$ is a F-vector space and $\varphi : V \times W \to V \otimes W$ is a bilinear map that satisfies the following universal property: For every F-vector space Z and any bilinear map $f : V \otimes W \to Z$ there is an unique linear map $\overline{f} : V \otimes W \to Z$ such that $f = \overline{f} \circ \varphi$

If A, B are F-algebras, we can make $A \otimes B$ into a F algebra by setting

$$(a \otimes b) \cdot (a' \otimes b') = (aa' \otimes bb')$$

for all $a, a' \in A$, $b, b' \in B$.

Given any vector space V, there is a special F-algebra $T(V)$ together a linear map $i : V \to T(V)$ with the following universal property: Given any F-algebra A and a linear map $f : V \to A$, there is a unique F-algebra homomorphism $\overline{f} : T(V) \to A$ so that $f = \overline{f} \circ i$.

The algebra $T(V)$ is called the **tensor algebra of** V and it may be constructed as the direct sum

$$T(V) = \bigoplus_{i \leq 0} V^{\oplus i}$$

where $V^0 = F$ and $V^{\oplus i}$ is the i-fold tensor product of V with itself for $i \geq 1$. There are natural injections $i_n : V^{\oplus n} \to T(V)$. Since any $v \in T(V)$ can be expressed as a finite sum

$$v = v_1 + \cdots v_n$$

where $v_i \in V^{\oplus n_i}$, and $n_i \neq n_j$ if $i \neq j$; to define multiplication in $T(V)$ we can use bilinearity, so it is enough to define the multiplication $V^{\oplus m} \times V^{\oplus n} \to V^{\oplus m+n}$ by

$$(u_1 \otimes \cdots \otimes u_m) \cdot (w_1 \otimes \cdots \otimes w_n) = u_1 \otimes \cdots \otimes u_m \otimes w_1 \otimes \cdots \otimes w_n$$

The multiplicative unit 1 in $T(V)$ is the image of $i_0(1)$ in $T(V)$ of the unit of the field F

Definition 1.3. Let V be a finite F-vector space and $B : V \times V \to F$ a symetric bilinear map. Consider $q : V \to F$ defined by $q(x) = B(x.x)$ The map q is a **quadratic form** and the pair (V, q) is a **quadratic space**. We say that an element $x \in V$ is **isotropic** if $B(x, x) = 0$, otherwise x is **anisotropic**

The following properties are easy to check:

- $q(ax) = a^2 q(x)$ for any $x \in V$ and $a \in F$.
- $q(x + y) - q(x) - q(y) = 2B(x, y)$ for any $x, y \in V$.

Since B and q determine each other, we can identify a quadratic space by (V, B). If (V, B) and (V', B') are quadratic space, we say that they are **isometric** if there is a vector space isomorphism $\tau : V \to V'$ such that $B(x, y) = B'(\tau(x), \tau(y))$ for any $x, y \in V$.

If we choose a coordinate basis for V, namely $\{e_1, \ldots, e_n\}$ then the quadratic space give rise to a quadratic form

$$f(x_1, \ldots, x_n) = \sum_{ij} B(e_i, e_j) x_i x_j = x^t M x$$

where $x = (x_1, \ldots, x_n)$ and $M_q = (B(e_i, e_j))_{ij}$ is a $n \times n$ symmetric matrix. Choosing another basis, we get a different matrix M'_q which satisfies the relation

$$M'_q = C^t M_q C$$

where C is the change of basis matrix. Therefore, a quadratic space (V, B) determines uniquely an equivalence classes of quadratic forms.

Given two quadratic spaces (V_1, B_1) and (V_2, B_2) of dimensions n and m respectively, we can construct a new quadratic space (V, B) by setting $V = V_1 \oplus V_2$ and $B((x_1, x_2), (y_1, y_2)) = B_1(x_1, y_1) + B_2(x_2, y_2)$. This space is called **the orthogonal sum** of V_1 and V_2, the associated quadratic form to B is denoted by $q_1 \perp q_2$ where q_i is the respective associated quadratic form to B_i, $i = 1, 2$. In this case V is a $n + m$ dimensional quadratic space.

On the other hand, by setting $W = V_1 \otimes V_2$ and $B(x_1 \otimes x_2, y_1 \otimes y_2) = B_1(x_1, y_1)B_2(x_2, y_2)$ we have also a new quadratic space. This space is called **the Kronecker Product** of V_1 and V_2. The associated quadratic form is $q = q_1 \cdot q_2$, in this case (W, B) is a mn dimensional quadratic space.

Theorem 1.4 (**Witt's Cancellation Theorem**). *If q, q_1, q_2 are arbitrary quadratic forms such that $q \perp q_1 \cong q \perp q_2$, then $q_1 \cong q_2$.*

□

For any $d \in F$, we shall denote $\langle d \rangle$ the isometry class of the $1-$dimensional space corresponding to the quadratic form dx^2. The next result allow us to diagonalize any quadratic form.

Proposition 1.5. If (V, B) is any quadratic space over F, then there exist scalars $d_1, \ldots, d_n \in F$ such that $V \cong \langle d_1 \rangle \oplus \cdots \oplus \langle d_n \rangle$; in other words, the quadratic form q associated to B is equivalent to some diagonal form $d_1x_1^2 + \cdots + d_nx_n^2$. By setting notation, we shall write $V = \langle d_1, \ldots d_n \rangle$, and the n-ary form by $\langle d, \ldots, d \rangle = n\langle d \rangle$. □

With this setting, the operations between quadratic spaces are easily computable; namely , if $q_1 = \langle a_1, \ldots, a_n \rangle$ and $q_2 = \langle b_1, \ldots, b_m \rangle$ we have

$$q_1 \perp q_2 = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$$

$$q_1 \cdot q_2 = \langle a_1b_1, a_1b_2, \ldots, a_nb_m \rangle$$

We discuss now the **determinant** of a non-singular quadratic form q. This is defined to be

$$d(q) = \det(M_q) \bmod (F^*)^2 \in F^*/(F^*)^2$$

Observe that if $q \cong q'$ then $M'_q = C^t M_q C$, and hence

$$d(q') = \det(M_q)(\det(C))^2 \equiv \det(M_q) \bmod (F^*)^2 = d(q)$$

Now, if $q = \langle d_1, \ldots, d_n \rangle$, then $d(q) = d_1 \cdots d_n$.

Definition 1.6. A **graded algebra** A is a finite dimensional F-algebra given in the form $A = A_0 \oplus A_1$, such that $F = F \cdot 1 \subseteq A_0$ and $A_iA_j = A_{i+j}$ where the subscripts are taken modulo 2.

If $a \in A$, we write $\partial(a) = i$ iff $a \in A_i$. We shall now introduce the **graded tensor product** of two graded algebras A, B, denoted by $A\hat{\otimes}B$; the i-component ($i = 0, 1$) is defined by

$$(A\hat{\otimes}B)_i = \sum_{1,j} A_j \otimes B_k \text{ where } j + k \equiv i \bmod 2$$

The multiplication on $A\hat{\otimes}B$ is given by

$$(a \otimes b)(a' \otimes b') = (-1)^{\partial b \partial a'} aa' \otimes bb'$$

Notice that $A\hat{\otimes}B$ is just the ordinary $A \otimes B$ as a vector spaces, so $\dim(A\hat{\otimes}B) = \dim(A)\dim(B)$.

Example 1.7. Define the **Graded Quaternion Algebra** $A = (\frac{a,b}{F})$ to be the algebra generated by $\{1, i, j, k\}$ satisfying the relations

$$i^2 = a, j^2 = b, k^2 = -ab$$

$$ik = -ki = aj, kj = -kj = bi$$

where $k = ij$. For an arbitrary quaternion $x = \alpha + \beta i + \gamma j + \delta k$, denote the conjugate of x by $\overline{x} = \alpha - (\beta i + \gamma j + \delta k)$, consider the quadratic form over A given by

$$N(x) = x\overline{x}$$

So (A, N) is a quadratic space, with orthogonal basis $\{1, i, j, k\}$.

To make A into a graded algebra, consider $A_0 = F \cdot 1 \oplus F \cdot k$ and $A_1 = F \cdot i \oplus F \cdot j$

Example 1.8. Consider the algebra $M_r(A)$ of $r \times r$ matrices with coefficients in A, where A is a graded algebra. Define the grading in $M_r(A)$ by setting

$$\widehat{M_r}(A)_0 = \begin{pmatrix} A_0 & A_1 & \\ A_1 & A_0 & \\ & & \ddots \end{pmatrix}, \widehat{M_r}(A)_1 = \begin{pmatrix} A_1 & A_0 & \\ A_0 & A_1 & \\ & & \ddots \end{pmatrix}$$

And we have the following graded algebra isomorphism

$$\widehat{M_r}(A) \cong \widehat{M_r}(F)\hat{\otimes}A$$

$$\widehat{M_r}(F)\hat{\otimes}\widehat{M_s}(A) \cong \widehat{M_{rs}}(A)$$

2 Construction of Clifford Algebras

In this section (V, q) will denote a general quadratic space

Definition 2.1. Let A be an F-algebra and $f : V \to A$ a injective linear map. (A, f) is said **compatible with** q if $f(x)^2 = q(x) \cdot 1$ for any $x \in V$.

We shall identify V as a subspace of A and $F \cdot 1$ with F, so the above equation becomes $x^2 = q(x)$. Notice that if B denotes the bilinear form on V associated with q, we get the equation

$$2B(x, y) = q(x + y) - q(x) - q(y) = (x + y)^2 - x^2 - y^2 = xy + yx$$

So, we have that x and y are **orthogonal in** V iff $xy = -yx$.

Lemma 2.2. *For A as above, and $x \in V$ non-zero, x is invertible in A iff x is an isotropic vector in V.*

Proof. Suppose that $xy = 1$ for $y \in A$. Then $q(x)y = x^2 y = x$, so $q(x) \neq 0$ since $x \neq 0$. Conversely, set $y = x/q(x)$, then $xy = x^2/q(x) = 1$. $\square$

Definition 2.3. Let (V, q) a quadratic space, a **Clifford Algebra** associated to (V, q) is a F algebra C together with a linear map $i : V \to C$ satisfying the condition $(i(v))^2 = q(v) \cdot 1$ for all $v \in V$, and so that has the following universal property:
For any $F-$algebra A and injective every linear map $f : V \to A$ such that (A, f) is compatible with q, there is a unique F-algebra homomorphism $\overline{f} : C \to A$ so that $f = \overline{f} \circ i$.

If we see V as a subspace of both C and A, we must have that $\overline{f}(v) = v$ for any $v \in V$. Notice that the universal property of C makes it unique up to isomorphism, so it is left to prove that for any given quadratic space (V, q) there is such associated Clifford Algebra.

Consider the tensor algebra $T(V)$ and let $I(q)$ the two sided ideal of $T(V)$ generated by elements of the form

$$x \otimes x - q(x) \cdot 1 \in T(V), \quad \text{where } \; x \in V$$

Define the quotient algebra $C(V) = T(V)/I(q)$, and the map $i : V \to C$ as the composition of the maps

$$V \xrightarrow{i_1} T(V) \xrightarrow{\pi} C(V)$$

where π denotes the quotient map. Observe that is not clear that i is an injection of V into $C(V)$, but we will prove this fact later, first we are going to show some examples

Example 2.4. Let $V = \langle a \rangle$ be the one-dimensional space with quadratic form $q(x) = a$ with basis $\{x\}$. In this case we may identify the algebra $T(V)$ with the polynomial ring $F[x]$ and $I(q)$ the ideal generated by $x^2 - a$. Thus, $C(V)$ is the quotient $F[x]/(x^2 - a)$. So $\{1, x\}$ is a basis for $C(V)$ and any element can be written in the form $\alpha + \beta x$, where $\alpha, \beta \in F$. The multiplication is given by

$$(\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 x) = (\alpha_1 \alpha_2 + a\beta_1 \beta_2) + (\alpha_1 \beta_2 + \alpha_2 \beta_1)x$$

Example 2.5. Let V be a 2-dimensional quadratic space with diagonalization $\langle a, b\rangle$, relative to the orthogonal basis $\{x, y\}$, let A the graded quaternion algebra $(\frac{a,b}{F})$ defined in example 1.7. We may embed V into A by identifying x, y with i, j respectively. Then

$$(\alpha x + \beta y)^2 = (\alpha i + \beta j)^2 = \alpha^2 a + \beta^2 b = q(\alpha x + \beta y)$$

so A is compatible with (V, q), moreover, A satisfies the universal property for the Clifford algebra associated to (V, q); therefore,

$$C(V, q) \cong A$$

Example 2.6. Denote by $\mathbb{H} = \langle 1, -1\rangle$ the hyperbolic space, the above example shows that $C(\mathbb{H}) \cong (\frac{1,-1}{F})$, and consider the map $\varphi : (\frac{1,-1}{F}) \to M_2(F)$ given by $\varphi(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \varphi(j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \varphi(k) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ This map is an algebra isomorphism; moreover, under this map, the image of $F \oplus F \cdot k$ correspond to the matrices $\begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix}$, and the image of $F \cdot i \oplus F \cdot j$ correspond to the matrices $\begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}$. So φ is a graded algebra isomorphism and thus $C(\mathbb{H}) \cong \widehat{M_2}(F)$.

For any Clifford Algebra $C(V)$, we have a canonical automorphism α, defined using the next proposition

Proposition 2.7. Every Clifford Algebra $C(V)$ has a unique automorphism $\alpha : C(V) \to C(V)$ satisfying the properties

$$\alpha \circ \alpha = id \text{ and } \alpha(i(v)) = -i(v)$$

Proof. Consider the linear map $\alpha_0 : V \to C(V)$ defined by $\alpha_0(v) = -i(v)$. Since $(C(V), \alpha_0)$ is compatible with q, we have an algebra homomorphism $\alpha : C(V) \to C(V)$ such that $\alpha \circ i = \alpha_0$. This implies that for any $x \in i(V)$, we have that $\alpha(x) = -x$; furthermore, every $x \in C(V)$ can be written as $x = x_1 \cdots x_m$ with $x_i \in (V)$, so $\alpha \circ \alpha(x) = x$.

□

Define $C_i(V) = \{x \in C(V) | \alpha(x) = (-1)^i x\}$, for $i = 0, 1$, we have that $C(V) = C_0(V) \oplus C_1(V)$ and $C_i C_j \subseteq C_{i+j}$ (the subscript here is taken modulo 2) so $C(V)$ has a structure of graded algebra. Observe that $C_0(V)$ is the image of $\bigoplus T^i(V)$ (i even) under the quotient map $\pi : T(V) \to C(V)$; similarly $C_1(V)$ is the image of $\bigoplus T^i(V)$ (i odd).

Remark: Observe that $C_0(V)$ is a subalgebra of $C(V)$, whereas $C_1(V)$ is not.

We shall compute the dimension of $C(V)$, first we need the following result

Lemma 2.8. *If (V, q) and (V', q') are quadratic spaces, there exist a surjection*

$$f : C(V \oplus V') \to C(V) \hat{\otimes} C(V')$$

Proof. Consider the map $\phi : V \oplus V' \to C(V)\hat{\otimes}C(V')$ given by $\phi(x, x') = x \otimes 1 + 1 \otimes x'$. Clearly, $ker(\phi) = \{(0, 0)\}$, so ϕ is an injective linear map. Furthermore,

$$\phi(x, x)^2 = (x \otimes 1 + 1 \otimes x')^2 =$$

$$x^2 \otimes 1 + 1 \otimes x'^2 + (x \otimes 1)(1 \otimes x') + (1 \otimes x')(x \otimes 1) =$$

$$x^2 \otimes 1 + 1 \otimes x'^2 + (x \otimes x') + (-1)(x \otimes x') = q(x) + q'(x') = (q \oplus q')(x, x')$$

So by the universal property of the Clifford Algebra, we have an unique algebra homomorphism

$$f : C(V \oplus V) \to C(V)\hat{\otimes}C(V')$$

which coincides with ϕ on $V \oplus V'$. Observe that f is surjective, since the elements of the form $x \otimes 1, 1 \otimes x'$ $(x \in V, x' \in V')$ lies in the image of f and $C(V)\hat{\otimes}C(V')$ is generated as F-algebra by this elements. □

Now we can prove the main result.

Theorem 2.9. *If (V, q) is a n-dimensional quadratic space, then* $\dim C(V) = 2^n$. *In particular, if $\{x_1, \ldots, x_n\}$ is an orthogonal basis for (V, q), then $\{x_1^{e_1} \cdots x_n^{e_n} : e_i = 0, 1\}$ constitutes a basis for $C(V)$.*

Proof. Let $\{x_1, \ldots, x_n\}$ be an orthogonal basis on V. Observe that for $i \neq j$, $x_i x_j = -x_j x_i$ since $0 = 2B(x_i, x_j) = x_i x_j + x_j x_i$, so the equations $x_i x_j = -x_j x_i$ and $x_i^2 = q(x_i)$ holds in $C(V)$. Thus, as an F space, $C(V)$ is spanned by the products of the form $x_1^{e_1} \cdots x_n^{e_n}, e_i = 0, 1$; therefore, $\dim C(V) \leq 2^n$.

We are going to prove the reverse inequality by induction and using the previous lemma. For $n = 1$, see example 2.4. For $n > 1$, take any orthogonal basis $\{x_1, \ldots, x_n\}$ for V, and set $U = span(x_1)$ and $U' = span(x_2, \ldots x_n)$. Since $V = U \oplus U'$, $C(V) = C(U \oplus U')$ which maps surjectively onto $C(U)\hat{\otimes}C(U')$; thus

$$\dim(C(V)) \geq \dim(C(U)) \dim(C(U'))$$

By the induction hypothesis, $\dim(C(U')) = 2^{n-1}$ and example 2.4 implies $\dim(C(U)) = 2$. So we have $\dim(C(V)) \geq 2^n$ completing the proof.

□

Consider the m-fold orthogonal sum $m\mathbb{H} = \mathbb{H} \oplus \cdots \oplus \mathbb{H}$, then using example 2.6 we have $C(m\mathbb{H}) \cong \widehat{M}_{2^n}(F)$. Any quadratic space that is isometric to $m\mathbb{H}$ for some $m \in \mathbb{N}$ is called **Hyperbolic Space**.

3 Introduction to Witt Rings and Brauer Group

Let M be a commutative cancellation monoid, define $\sim$ on $M \times M$ by

$$(x, y) \sim (x', y') \Leftrightarrow x + y' = x' + y$$

The cancellation law in M implies that $\sim$ is an equivalence relation on $M \times M$. Define the **Grothendieck group** of M to be $G(M) = (M \times M)/\sim$. So $G(M)$ is a group and the map $i : M \to G(M)$ given by $i(x) = [(x, 0)]$ is an injection. Since the inverse of $[(x, y)]$ is $[(y, x)]$, we have that $[(x, y)] = i(x) - i(y)$ can be identified with $x - y \in M$. Any monoid homomorphism $h : M \to G$ where G is an abelian group, can be extended uniquely to a group homomorphism $H : G(M) \to G$ by $H(x, y) = h(x) - h(y)$. Lastly, if M has a commutative multiplication, then it induces a commutative multiplication on $G(M)$ which makes it into a ring.

$$(x, y)(x', y') = (xx' + yy', yx' + xy')$$

We may apply this construction to quadratic forms. Consider $M(F)$ the set of all isometry classes of quadratic forms over the field F, the binary operations $\perp$ and $\otimes$ define the structure of a semi commutative ring on $M(F)$. By Witt's cancellation theorem, the operation $\perp$ makes $M(F)$ into a cancellation monoid.

Definition 3.1. The group $\widehat{W}(F) = G(M(F))$ is called the **Witt-Grothendieck ring** of quadratic forms over the field F.

Consider the dimension map $\dim : M(F) \to \mathbb{Z}$, this extends uniquely to a ring homomorphism $\widehat{W}(F) \to \mathbb{Z}$. The kernel of this homomorphism, denoted by $\hat{I}F$ is called the **fundamental ideal** of $\widehat{W}(F)$.

Definition 3.2. Consider the ideal that consist of all hyperbolic spaces and their additive inverses in $\widehat{W}(F)$, namely $\mathbb{Z} \cdot \mathbb{H}$. The factor ring $W(F) = \widehat{W}(F)/\mathbb{Z} \cdot \mathbb{H}$ is called the **Witt ring** of F.

The image of the ideal $\hat{I}F$ under the natural projection $\widehat{W}(F) \to W(F)$ is denoted by IF, this is called the **fundamental ideal** of $W(F)$.

Proposition 3.3.

- $\hat{I}F$ is additevly generated by the expressions $\langle a \rangle - \langle 1 \rangle$, where $a \neq 0$.
- $\hat{I}F \cong IF$.
- A form q represents an element in IF iff $\dim q$ is even.
- IF is generated by the forms $\langle 1, -a \rangle$, $a \in F^*$.
- $IF/I^2F \cong F^*/(F^*)^2$ where F^* is the multiplicative group $F \setminus \{0\}$.
- I^2F consist of classes of even-dimensional forms q for which $d(q) = (-1)^{n(n-1)/2}$ where $\dim(q) = n$.

□

Now we shall construct other algebraic object associated to F, it is called **The Brauer Group**. In the following, any F-algebra will always mean a finite dimensional F-algebra. For any subset S of an F-algebra A we shall write

$$C_A(S) = \{a \in A : as = sa \text{ for all } s \in S\}$$

which is called the **centralizer** of S in A, this is a subalgebra of A; for the special case $S = A$, we use the notation $C_A(A) = Z(A)$ called the **center** of A.

Definition 3.4. A is called F**-central** if $Z(A) = F \cdot 1$, A is called **simple** if A has no two-sided ideals other than (0) and A. If A satisfies both conditions, it is called **cental simple algebra** (CSA) over A.

The basic examples of CSA over a field F are $M_n(F)$ and the four dimensional quaternion algebra $(\frac{a,b}{F})$ where $a, b \neq 0 \in F$.

Theorem 3.5. *Let A, B F-algebras and $A' \subseteq A$, $B' \subseteq B$. Then*

$$C_{A\otimes B}(A' \otimes B') = C_A(A') \otimes C_B(B')$$

In particular, if A, B are both CSA over F, so is $A \otimes B$. □

Let A, A' both be CSAs over F. We say that A is **similar** to A' if there exist finite-dimensional vector spaces V, V' such that $A \otimes \operatorname{End} V \cong A' \otimes \operatorname{End} V'$. For instance, if $A = F$ and $A' = M_n(F)$, take $V = F^n$ and $V' = (0)$, then F is similar to $M_n(F)$.

Proposition 3.6. The relation of similarity defined above is an equivalence relation.

Proof. The reflexivity and symmetry are immediate by definition. To show that it has the transitive property, we need to use the fact that $(\operatorname{End} V) \otimes (\operatorname{End} V') \cong \operatorname{End}(V \otimes V')$. □

The equivalence class of A will be denoted by $[A]$, and we have that the operation

$$[A] \cdot [B] = [A \otimes B]$$

is well defined and makes the set of similarity classes of CSAs into a commutative monoid with $[F] = [M_n(F)]$. We denote this monoid by $B(F)$.

Denote by $A^{op} = \{a^{op} : a \in A\}$ the opposite algebra of A, where $(a^{op} \cdot b^{op} = (ba)^{op}$. Clearly, if A is CSA, so is A^{op}. Now define $\theta : A \otimes A^{op} \to \operatorname{End}(A)$ by setting

$$\theta(a \otimes b^{op})(c) = acb$$

θ is an algebra homomorphism and it is injective since $(A \otimes A^{op})$ is a CSA. So θ is an isomorphism by counting dimensions. Now we are allowed to give the following definition.

Definition 3.7. The **Brauer Group** $B(F)$ of F is the set of similar equivalence classes of CSAs over F. In particular $B(F)$ is an abelian group.

A characterization of this group can be realized using the Wedderburn Theorem.

Theorem 3.8 (**Wedderburn Theorem**)**.** *Every central simple algebra A over F is of the form $M_n(D)$, where D is a central division algebra over F, and D is uniquely determined up to isomorphism.*

Proof. □

Thus we can conclude

Corollary 3.9. *The elements in the Brauer Group $B(F)$ are in $1-1$ correspondence with the isomorphism classes of F-central division algebras.*

Proof. Let A be a CSA over F, by Wedderburn Theorem,

$$A \cong M_n(D) \cong M_n(F) \otimes D$$

where D is a division CSA; therefore, $[A] = [D]$ in $B(F)$. □

4 Properties of Quaternion Algebras

The Quaternion algebras introduced barely in example 1.7 play a vital role in quadratic form theory; in this section we present important properties of these 4-dimensional forms that will be useful for our purposes in the next sections.

Proposition 4.1.
1. The quaternion algebra $(\frac{a,b}{F})$ is either a Division algebra or isomorphic to the algebra $M_2(F)$.
2. $(\frac{a,b}{F})$ is a division algebra if and only if for any element $q \in (\frac{a,b}{F})$, $N(q) = 0 \Rightarrow q = 0$.
3. $(\frac{a,b}{F})$ and $(\frac{a',b'}{F})$ are isomorphic as F-algebras if and only if they are isometric as quadratic spaces.
4. For $a, b, c, d \in F^*$ we have
$$(\frac{a,b}{F}) \otimes (\frac{a,c}{F}) \cong (\frac{a,bc}{F}) \otimes (\frac{c,-a^2c}{F})$$

Proof.

1. By the Wedderburn Theorem, $(\frac{a,b}{F}) \cong M_n(D)$ for some division central algebra D. So
$$4 = \dim(\frac{a,b}{F}) = \dim(M_n(D)) = n^2 \dim(D)$$
Then we have either $n = 1$ or $n = 2$. if $n = 1$, then $\dim D = 4$ and so $D \cong (\frac{a,b}{F})$. If $n = 2$, $\dim D = 1$ and so $D \cong F$.
2. This is a consequence of lemma 2.1.
3. It is immediate that any isomorphism of algebras induces an isomorphism of quadratic spaces. For the converse, observe that any isometry is an isomorphism over the three dimensional subalgebra generated by $\{i, j, k\}$. Thus by the Witt's Cancellation Theorem, we get the result.

4. Let $\{1, i, j, k\}$ and $\{1, i', j', k'\}$ basis for $A = (\frac{a,b}{F})$ and $B = (\frac{a,c}{F})$ respectively. Consider the algebra spanned by

$$\begin{aligned} X &= F \cdot (1 \otimes 1) + F \cdot (i \otimes 1) + F \cdot (j \otimes j') + F \cdot (k \otimes j') \\ &= F \cdot 1 + F \cdot I + F \cdot J + F \cdot IJ \end{aligned}$$

Where $I = i \otimes 1,\ J = j \otimes j'$ and $IJ = k \otimes j'$. This is a four dimensional subalgebra of $A \otimes B$, and we have that

$$I^2 = i^2 \otimes 1 = a, \ \ J^2 = j^2 \otimes j'^2 = bc,$$

$$IJ = ij \otimes j' = -ji \otimes j' = -JI$$

So this subalgebra is isomorphic to $(\frac{a,bc}{F})$. Now consider the algebra spanned by

$$\begin{aligned} Y &= F \cdot (1 \otimes 1) + F \cdot (1 \otimes j') + F \cdot (i \otimes k') + F \cdot (-ci \otimes i') \\ &= F \cdot 1 + F \cdot \tilde{I} + F \cdot \tilde{J} + F \cdot \tilde{I}\tilde{J} \end{aligned}$$

This is a four dimensional subalgebra of $A \otimes B$, and we have that

$$\tilde{I}^2 = 1 \otimes j'^2 = c, \ \ \tilde{J}^2 = i^2 \otimes k'^2 = -a^2 c,$$

$$\tilde{I}\tilde{J} = i \otimes j'k' = i \otimes -k'j' = -\tilde{J}\tilde{I}$$

So this subalgebra is isomorphic to $(\frac{c,-a^2c}{F})$ Since each element of $\{I, J\}$ commutes with any element of $\{\tilde{I}, \tilde{J}\}$, elements of X commute with elements of Y as well; so by counting dimensions we have

$$A \otimes B \cong X \otimes Y \cong (\frac{a, bc}{F}) \otimes (\frac{c, -a^2 c}{F})$$

as desired.

□

Now we can state the following useful results for quaternion algebras as corollary

Corollary 4.2. *For any* $a, b, x, y \in F^*$*,*

1. $(\frac{a,b}{F}) \cong (\frac{ax^2,by^2}{F})$
2. $(\frac{a,b}{F}) \cong (\frac{b,a}{F})$.
3. $(\frac{a,a}{F}) \cong (\frac{a,-1}{F})$.

4. $(\frac{a,-a}{F}) \cong (\frac{1,a}{F}) \cong M_2(F)$.

5. If $a \neq 1$, $(\frac{a,1-a}{F}) \cong M_2(F)$.

6. $(\frac{a,b}{F}) \otimes (\frac{a,x}{F}) \cong (\frac{a,bx}{F}) \otimes M_2(F)$.

□

Now we introduce another notion on quaternion algebras and field extensions.

Definition 4.3. Let $F \subseteq L$ a field extension and A quaternion algebra over F. We say that F is **split over** L if $A \otimes_F L$ is isomorphic to a matrix algebra over F. We just say that A is **split** if it is split over F.

The following is immediate from the previous results

- if $a + b = 1$ then $(\frac{a,b}{F})$ is split.
- $(\frac{a,b}{F}) \otimes (\frac{c,d}{F})$ is split if and only if $(\frac{a,b}{F}) \cong (\frac{c,d}{F})$.

5 Real Periodicity and Clifford Modules

Let F any field, let $\varphi_{p,q}$ denote the form $p\langle -1\rangle \perp q\langle 1\rangle$. We shall write $C^{p,q} = C(V, \varphi_{p,q})$ with the convention $C^{0,0} = F$.

Proposition 5.1. There is a graded algebra isomorphism

$$C^{p+n,q+n} \cong \widehat{M}_{2^n}(C^{p,q})$$

Proof. Observe that $\varphi_{p+n,q+n} \cong \varphi_{p,q} \perp \varphi_{n,n}$, thus by lemma 2.8

$$C^{p+n,q+n} \cong C^{p,q} \hat{\otimes} C^{n,n} \cong C^{p,q} \hat{\otimes} \widehat{M}_{2^n}(F) \cong \widehat{M}_{2^n}(C^{p,q})$$

□

So, we only need to calculate $C^{p,0}$ and $C^{0,q}$. By the following proposition we get a second reduction

Proposition 5.2 (Periodicity 8**).** $C^{p+8,q} \cong \widehat{M}_{16}(C^{p,q}) \cong C^{p,q+8}$.

Proof. Consider first $p = q = 0$, observe that $C^{0,4} \cong C^{4,0}$.

$$C^{8,0} \cong C^{4,0} \hat{\otimes} C^{4,0} \cong C^{4,0} \hat{\otimes} C^{0,4} \cong C^{4,4} \cong \widehat{M}_{16(F)}$$

This implies

$$C^{p+8,q} \cong C^{p,q} \hat{\otimes} C^{8,0} \cong C^{p,q} \hat{\otimes} \widehat{M}_{16}(F) \cong \widehat{M}_{16}(C^{p,q})$$

Similarly, $C^{0,8} \cong \widehat{M}_{2^n}(C^{p,q})$. □

So we only need to know $C^{p,0}$ and $C^{0,q}$ for $0 \le p,q \le 7$. These are computed in terms of $X = C^{1,0} \cong F(\sqrt{-1}), Y = C^{2,0} \cong (\frac{-1,-1}{F})Z = C^{0,1} \cong F(\sqrt{1}), W = C^{0,2} \cong (\frac{1,1}{F})$. The other Clifford Algebras are easily computed and can be found in the following chart

n	0	1	2	3	4	5	6	7
$C^{n,0}$	F	X	Y	$Y \otimes Z$	$Y \otimes W$	$\widehat{M_2}(X \otimes W)$	$\widehat{M_4}(W)$	$\widehat{M_8}(Z)$
$C^{0,n}$	F	Z	W	$X \otimes W$	$Y \otimes W$	$\widehat{M}_(Y \otimes Z)$	$\widehat{M_4}(Y)$	$\widehat{M_8}(X)$

- Since $C^{3,0} \cong C^{2,0} \hat{\otimes} C^{1,0} \cong C^{2,0} \otimes C^{0,1} \cong Y \otimes Z$.
- Similarly, $C^{4,0} \cong C^{2,0} \otimes C^{0,2} \cong Y \otimes W$
  $C^{0,3} \cong C^{0,2} \otimes C^{1,0} \cong W \otimes X$
  $C^{0,3} \cong C^{0,2} \otimes C^{2,0} \cong W \otimes Y$
- Finally, for $p = 1,2,3$ we have

$$C^{p+4,0} \cong C^{p,0} \hat{\otimes} C^{4,0} \cong C^{p,0} \hat{\otimes} C^{0,4} \cong C^{p,4} \cong \widehat{M}_{2^p}(C^{0,4-p})$$

  by 5.1.
- The three remaining Clifford Algebras $C^{0,5}, C^{0,6}$ and $C^{0,7}$ are computed similarly.

6 Steinberg Symbols and Milnor's Group k_2F

In algebraic K-theory, a series of K-groups denoted by K_nR is associated to any ring R. For a ring R, let K_0R be the Grothendieck group of finitely generated projective modules over R, and K_1R be the Bass-Whithead group R. if R is a field F, can be proved that $K_0F \cong \mathbb{Z}$ and $K_1F \cong F^*$, these *may be taken to be the definitions* for the first two K-groups of the field F. In particular, if we define $k_n := K_nF/2K_nF$, then by lemma 3.3

$$k_0F \cong \mathbb{Z}_2 \cong W(F)/IF, k_1F \cong F^*/(F^*)^2 \cong IF/I^2F$$

So it would be natural to ask if the next filtration factor I^2F/I^3F of the Witt ring is can be also described from the viewpoint of the algebraic K-theory of fields.

We shall define the group K_2F following Milnor's idea and show that I^2F/I^3F is isomorphic to the group of k_2F.

Definition 6.1. A pairing $f : F^* \times F^* \to G$ into a multiplicative abelian group F is said to be a **Steinberg Symbol** if f is bimultiplicatibe and $f(a,b) = 1$ whenever $a + b = 1$, this property will be referred to as the **Steinberg property**.

In view of the universal property of the tensor prodcut $F^* \otimes F^*$, we can construct a **universal Steinberg Symbol** as the quotient

$$F^* \otimes F^* / \langle a \otimes b : a + b = 1 \rangle$$

which is defined to be K_2F. The natural pairing $\varphi : F^* \times F^* \to K_2F$ is a Steinberg symbol with the universal property that for any Steinberg symbol $f : F^* \to F^* \to G$ exist an unique group homomorphism $g : K_2F \to G$ such that $f = g \circ \varphi$.

Write $[a, b]$ for the image of $a \otimes b$ in K_2F. We have the following properties in K_2F.

Proposition 6.2.

1. If $a + b = 0$ then $[a, b] = 1$.
2. $[a, b] = [b, a]^{-1}$.
3. $[a, a] = [a, -1]$ and has order ≤ 2.
4. If $a + b \neq 0$, then $[a.b] = [a + b, -b/a]$.

Proof. 1. We may assume that $a \neq 1$, so $1 - a^{-1} \neq 0$. Using

$$1 = [a^{-1}, 1 - a^{-1}] = [a, 1 - a^{-1}]^{-1}$$

we have

$$[a, -a] = [a, -a][a, 1 - a^{-1}] = [a, -a + 1] = 1$$

2. From (1) we get

$$1 = [ab, -ab] = [a, -a][a, b][b, a][b, -b] = [a, b][b, a]$$

3. Observe that

$$1 = [a, -a] = [a, a][a, -1]$$

so $[a, -1] = [a.a][a, -1]^2 = [a, a][a, 1] = [a, a]$.

4. Letting $c = a + b \neq 0$, we have $ac^{-1} + bc^{-1} = 1$. So

$$1 = [ac^{-1}, bc^{-1}] = [a, b][c, b]^{-1}[a, c]^{-1}[c, c]$$

implies

$$[a, b] = [c, b][a, c][c, -1] = [c, b][c, a^{-1}][c, -1] = [c, -b/a]$$

□

Recall that $k_2F = K_2F/(K_2F)^2$, we shall write again $[a, b]$ for the image of $[a, b]$ in k_2. Additionally, we have the following immediate properties in k_2 as consequence of proposition 6.2

- For any $a, b, c \in F^*$, $[ac^2, b] = [a, bc^2] = [a, b]$.
- For any $a, b \in F^*$, $[a, b] = [b, a]$

Lemma 6.3. *The following relations are true in $k_2(F)$*

1. $[a, x^2 - ay^2] = 0$ *for all* $a \in F$, $x, y \in F^*$ *satisfying* $x^2 - ay^2 \neq 0$.

2. $[a, b] = [ab, ab(a+b)]$ *for all* $a, b \in F^*$ *satisfying* $a + b \neq 0$.

Proof. 1. If $x = 0$ the result follows immediately from the previous remark. If $x \neq 0$ consider the relations

$$0 = [a(yx^{-1})^2, 1 - a(yx^{-1})^2] = [a, x^2 - ay^2]$$

2. Since $(a+b)^2 = a(a+b) + b(a+b)$ by (1)

$$[a(a+b), b(a+b)] = [a, b] + [a+b, ab(a+b)]$$

□

Proposition 6.4. Let ${}_2B(F) := \{x \in B(F) : x^2 = 1\}$ where $B(F)$ is the Brauer Group of F. Then

$$(a, b) \mapsto (\frac{a, b}{F}) \in {}_2B(F)$$

is a Steinberg symbol . This symbol is induced by an unique group homomorphism $\beta : k_2F \to {}_2B(F)$ given by

$$\beta[a, b] = (\frac{a, b}{F})$$

Proof. First we have to prove that the map is indeed well-defined. By corollary 4.2

$$(\frac{a, b}{F}) \otimes (\frac{a, b}{F}) \cong (\frac{a, b^2}{F}) \otimes M_2(F) \cong (\frac{a, 1}{F}) \otimes M_2(F)$$

$$\cong M_2(F) \otimes M_2(F) \cong M_4(F)$$

Therefore $[(\frac{a,b}{F})]^2 = 1$. The bi-multiplicative property in a, b follows again from corollary 4.2. Observe that $\beta(\frac{a,1-a}{F}) \cong M_2(F)$ and hence $[(\frac{a,1-a}{F})] = 1$. So β induces a Steinberg symbol as desired. □

Proposition 6.5. Recall that $W(F)$ denotes the classes of anisotropic quadratic forms over F and $I(F)$ the ideal generated by the forms $\langle 1, -a\rangle$. The map

$$(a, b) \mapsto \langle 1, -a\rangle\langle 1, -b\rangle + I^3F$$

is a Steinberg symbol into I^2F/I^3F. This symbol is induced by an unique group homomorphism $\alpha : k_2F \to I^2F/I^3F$ given by

$$\alpha[a, b] = \langle 1, -a\rangle\langle 1, -b\rangle + I^3F$$

Proof. First we shall prove that the map is well defined. If $a + b = 1$, then

$$\langle 1, -a\rangle\langle 1, -b\rangle \cong (\frac{a, b}{F}) \cong M_2(F) = 0 \in W(F).$$

So we just have to prove that it is bimultiplicative. In fact,

$$\begin{aligned}
\langle 1,-a\rangle\langle 1,-b\rangle + \langle 1,-a'\rangle\langle 1,-b\rangle &= \langle 1,-a,-b,ab\rangle + \langle 1,-a',-b,a'b\rangle \\
&= \langle 1,-a,-b,ab,1,-a',-b,a'b\rangle \\
&= \langle 1,-a,1,-a'\rangle\langle 1,-b\rangle \\
&= (\langle 1,-a,1,-a'\rangle + \langle aa',-aa'\rangle)\langle 1,-b\rangle \qquad (\langle aa',-aa\rangle = M_2(F) = 0 \in W(F)] \\
&= \langle 1,-a,-a',aa',1,-aa'\rangle\langle 1,-b\rangle \\
&= (\langle 1,-a\rangle\langle 1,-a'\rangle + \langle 1,-aa'\rangle)\langle 1,-b\rangle \\
&\equiv \langle 1,-aa'\rangle\langle 1,-b\rangle (\mathrm{mod}\ I^3F)
\end{aligned}$$

□

Now we want to relate the groups ${}_2B(W)$ and I^2F/I^3F.
Since I^2F is generated by the forms $\langle 1,-a\rangle\langle 1,-b\rangle$, we define a homomorphism γ from I^2 into ${}_2B(W)$ by

$$\langle 1,-a\rangle\langle 1,-b\rangle \mapsto (\frac{a,b}{F})$$

that is well defined using a similar argument that in proof of proposition 6.4. We have as immediate properties:

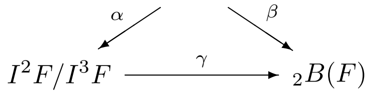
- $\gamma(I^3F) = 1$.
- $\gamma(I^2F) = \mathrm{Quat}(F)$, where $\mathrm{Quat}(F)$ is the subgroup of ${}_2B(F)$ generated by all quaternion algebras over F.

The second conclusion just follow from the definition of γ. For the first statement, recall that I^3F is additively generated by elements of the form

$$\begin{aligned}
q &= \langle 1,-a\rangle\langle 1,-b\rangle\langle 1,-c\rangle \\
&= \langle 1,-a,-b,ab\rangle\langle 1,-c\rangle \\
&= \langle 1,-a,-b,ab,-c,ac,cb,-abc\rangle \\
&= \langle 1,-a,-b,ab\rangle - \langle c,-ac,-bc,abc\rangle \\
&= \langle 1,-a\rangle\langle 1,-b\rangle - c^2\langle 1,-a\rangle\langle 1,-b\rangle
\end{aligned}$$

Applying γ and using that $(\frac{a,b}{F}) = (\frac{c^2a,c^2b}{F})$ we get $\gamma(q) = 1$.

Therefore, we can see γ as a homomorphism $\gamma : I^2F/I^3F \to {}_2B(F)$, and combining with propositions 6.4 and 6.5 we have a commutative diagram



In this connection, we have that α, β, γ are indeed isomorphism. We shall present some remarks for the proof of this fact based on Milnor and Merkurjev theorems. First, Milnor [1970] proved that α is an isomorphism; then the next major project was to deal with β and γ when finally in [1981] Merkurjev proved that β is an isomorphism.

Theorem 6.6 (Milnor, 1970)**.** *The map α is an isomorphism*

Proof. The idea is construct an inverse for α, following Milnor, we define the **Stiefel-Whitney class** of a quadratic form $q = \langle a_1, \ldots, a_n \rangle$ to be

$$w(q) = \prod_{i<j} [a_i, a_j] \in k_2 F$$

- $w(q)$ is an invariant of the quadratic form, that is, it depends only on the isometry class of q.

  Let q_1, q_2 isometric quadratic forms, that is $q_1 = \langle a_1, \ldots, a_n \rangle \cong q_2 = \langle b_1, \ldots b_2 \rangle$. We assert that there are two indexes i, j such that $\langle a_i, a_j \rangle \cong \langle b_i, b_j \rangle$ and $a_k = b_k$ whenever $k \neq i, j$. We write then $q_1 \sim q_2$.

  By induction on n: Suppose $n \leq 3$ (there is nothing to prove for $n = 1, 2$). Let $f = \langle c_1, \ldots, c_n \rangle$ such that $b_1 = c_1 e_1^2 + \cdots c_p e_p^2$, and p is the smallest possible and $f \sim q_1$. This such f exists by the Well-Ordering Principle. We claim that $p = 1$, suppose the contrary; by the minimality of p, no subsum can be equal to zero, so consider $d = c_1 e_1^2 + c_2 e_2^2$. Then $\langle c_1, c_2 \rangle \cong \langle d, c_1 c_2 d \rangle$ implies

$$\begin{aligned} q_1 \sim f &= \langle c_1, c_2, \ldots, c_n \rangle \\ &\cong \langle d_1, c_1 c_2 d, \ldots, c_n \rangle \\ &\cong \langle d, c_3, \ldots, c_n, c_1 c_2 d \rangle \end{aligned}$$

  and $b_1 = d + c_3 e_3^2 + \cdots c_p e_p^2$ which is a contradiction with the minimality of p. Thus $p = 1$ and hence $\langle c_1 \rangle \cong \langle b_1 \rangle$ and so $q_1 \cong \langle b_1, c_2, \ldots, c_n \rangle$. By Witt's Cancellation Theorem,

$$\langle c_2, \ldots, c_n \rangle \cong \langle b_2, \ldots b_n \rangle$$

  By the induction hypothesis, we get $\langle c_2, \ldots, c_n \rangle \sim \langle b_2, \ldots b_n \rangle$. Finally we get

$$q_1 \sim \langle c_1, c_2 \ldots, c_n \rangle \cong \langle b_1, c_2 \ldots, c_n \rangle \sim \langle b_1, b_2 \ldots, b_n \rangle = q_2$$

- This latter proof shows that the invariance of $w(q)$ depends only in the case of binary forms; that is, if $\langle a, b \rangle \cong \langle c, d \rangle$ then $[a, b] = [c, d]$.

  This can be shown as follows; write $c = ax^2 + by^2$, where $x, y \in F$. If $y = 0$, then $c = ax^2$, and the determinants of the quadratic form implies that $d = bz^2$ for some $z \in F^*$, since

$$ab = cd \mod (F^*)^2 \Rightarrow ab = ax^2 d(w^2) \Rightarrow d = bz^2$$

Thus $[c,d] = [ax^2, bz^2] = [a,b]$. Now if both $x, y \neq 0$, then we have by ...

$$[a,b] = [ax^2, by^2] = [ax^2 + by^2, -by^2/ax^2] = [c, -ab(y/ax)^2]$$
$$= [c, -ab] = [c, -cd] = [c, d]$$

- Observe that $w(q_1 \perp q_2) = w(q_1)(q_2)[d(q_1), d(q_2)]$ since

$$w(q_1 \perp q_2) = w(\langle a_1, \dots, a_n, b_1, \dots, b_n \rangle) = \prod_{i<j} [a_i, a_j] \prod_{i,j} [b_i, b_j] \prod_{i,j=1}^{n} [a_i, b, j]$$
$$= w(q_1)w(q_2)[d(q_1)^n, d(q_2)^n] = w(q_1)w(q_2)[d(q_1), d(q_2)]$$

- If we take $q_1, q_2 \in IF^2$, then $d(q_i) = (-1)^{m_i}$ for $m_i = (\dim q_i)/2$, so

$$w(q_1 \perp q_2) = w(q_1)w(q_2)[-1,-1]^{m_1 m_2}$$

  So in order to make w multiplicative and eliminate the error altogether, we define a signed Stiefel-Whitney Class. For any form of dimension $n = 2m$ consider

$$w_{\pm}(q) = w(q)[-1,-1]^{m(m-1)/2} = \begin{cases} w(q) & \text{if } n \equiv 0, 2 \bmod 8 \\ w(q)[-1,1] & \text{if } n \equiv 4, 6 \bmod 8 \end{cases}$$

- $w_{\pm}$ gives a well-defined group homomorphism from I^2F to k_2F.

$$\begin{aligned} w_{\pm}(q_1)w_{\pm}(q_2) &= w(q_1)w(q_2)[-1,-1]^{m_1(m_1-1)/2}[-1,-1]^{m_2(m_2-1)/2} \\ &= w(q_1)w(q_2)[-1,-1]^{m_1 m_2}[-1,-1]^{(2m_1 m_2 + m_1(m_1-1) + m_2(m_2-1))/2} \\ &= w(q_1 \perp q_2)[-1,-1]^{(m_1+m_2)(m_1+m_2-1)/2} \\ &= w_{\pm}(q_1 \perp q_2) \end{aligned}$$

- $w_{\pm}(\langle 1, a \rangle \langle 1, b \rangle) = [-a, -b] \in k_2F$.

  Since $\langle 1, a \rangle \langle 1, b \rangle = \langle 1, a, b, ab \rangle$. Thus

$$\begin{aligned} w_{\pm}(\langle 1, a \rangle \langle 1, b \rangle) &= [1,a][1,b][1,ab][a,b][a,ab][b,ab][-1,-1] \\ &= [1,ab]^2[a,b][ab,ab][-1,-1] \\ &= [a,b][ab,-1][-1,-1] \\ &= [a,b][a,-1][b,-1][-1,-1] \\ &= [a,b][a,-1][-b,-1] \\ &= [a,-b][-b,-b] \\ &= [-ab,-b] \\ &= [-a,-b][b,-b] = [-a,-b] \end{aligned}$$

- $w_\pm(I^3F) = 1$
  For any 4 dimensional form $q = \langle a_1, a_2, a_3, a_4$ which $d(q) = 1$, and any $c \in F^*$ we have that

$$\begin{aligned} w(\langle c\rangle q) &= [ca_1, ca_2][ca_1, ca_3][ca_1, ca_4][ca_2, ca_3][ca_2, ca_3][ca_3, ca_4] \\ &= [ca_1, a_2][ca_1, a_3][ca_1, a_4][ca_2, a_3][ca_2, a_3][ca_3, a_4] \\ &= [c, d(q)][a_1, a_2][a_1, a_3][a_1, a_4][a_2, a_3][a_2, a_4][a_3, a_4] \\ &= [a_1, a_2][a_1, a_3][a_1, a_4][a_2, a_3][a_2, a_4][a_3, a_4] \\ &= w(q) \end{aligned}$$

  Therefore, for elements in IF^3 we have

$$\begin{aligned} w_\pm(\langle 1, a\rangle\langle 1, b\rangle\langle 1, c\rangle) &= w_\pm(\langle 1, a\rangle\langle 1, b\rangle \perp \langle c\rangle\langle 1, a\rangle\langle 1, b\rangle) \\ &= w_\pm(\langle 1, a\rangle\langle 1, b\rangle)w_\pm(\langle 1, a\rangle\langle 1, b\rangle) \\ &= 1 \end{aligned}$$

In conclusion, $w_\pm : I^2F/I^3F \to k_2F$ defines a group homomorphism and checking generators, $w_\pm$ is an inverse to α. □

Now we focus in proving that the homomorphism $\beta : k_2(F) \to {}_2B(F)$ is an isomorphism.

Theorem 6.7 (Merkurjev, 1981)**.** *The map β is an isomorphism*

Proof. We prove that the map is injective and surjective.

- **Conic Curves in Quaternion Algebras**: In a quaternion algebra Q, consider the maps $T : Q \to Q$ and $N : Q \to Q$ given by $T(a) = a + \overline{a}$ and $N(a) = a\overline{a}$. Every element $a \in Q$ satisfies

$$a^2 - T(a)a + N(a) = 0$$

Set $V_Q := Ker(T)$ is a 3-dimensional subspace of Q. Note that for any $x \in V_Q$, $x^2 = -N(x)$ and the map $\varphi_Q(x) = x^2$ is a quadratic form over V_Q.

The quadric C_Q of the quadratic form φ_Q in the projective plane is a smooth curve. Since $Q = (\frac{a,b}{F})$, $V_Q = Fi \oplus Fj \oplus Fk$ and C_Q is given by the equation

$$at_1^2 + bt_2^2 - abt_3^2$$

Consider the function field $F(C)$ the generic splitting field of the equation of C_Q, then Q is split over $F(C)$.

- The following conditions are equivalent

  1. Q is split
  2. C_Q is isomorphic to the projective line $\mathbb{P}^1$.
  3. C_Q has a rational point.

  And thus follows,

  1. Every divisor of C_Q of degree zero is principal
  2. If Q is a division algebra, the degree of every closed point is even.

- Recall that a closed point refer to a discrete valuation ring of $F(C)$, and a divisor of C_Q is an element of the free abelian group generated by the set of closed points, denote this set by $\text{div}(C)$. For any $D \in \text{div}(C)$ we have

$$D = \sum_p \text{ord}_p(D)p$$

where the sum is over all the closed points and $\text{ord}_p(D)$ are integers all zero except for a finite number. The degree of D is defined by the formula

$$\deg(D) = \sum_p \text{ord}_p(D)d_p$$

where d_p is the degree of the residue field $F(C)_p$ over $F(C)$.

- The residue homomorphism: For every closed point $x \in C$, there is a homomorphism
$$\partial_x : K_2(F(C)) \to K_1(F(x)) = F(x)^*$$
induced by the discrete valuation of the local ring $\mathcal{O}_{C,x}$. Moreover, we have a exact sequence
$$K_2(F) \to K_2(F(C)) \xrightarrow{\partial} \coprod_{x \in C} F(x)^* \to F^*$$

- Fix a closed point $x_0 \in C$ and for any $n \in \mathbb{Z}$, let L_n be the subspace
$$L_n = \{f \in F(C)^* | \mathrm{div}(f) + nx_0 \geq 0\} \cup \{0\}$$
Since any zero divisor is principal, for every point $x \in C$ of degree $2n$, we can choose a function $f_x \in L^*$ satisfying $\mathrm{div}(f_x) = x - nx_0$.

- **Restriction homomorphism:** Let L/F a field extension, then the field homomorphism inclusion $F \to L$ induces a *restriction ring homomorphism*
$$r_{L/F} : K_2(F) \to K_2(L)$$
taking a symbol $[a_1, a_2]_F$ to the symbol $[a_1, a_2]_L$. The image of $r_{L/F}(\alpha)$ is denoted by α_L for an element $\alpha \in K_2(F)$.

  In particular, If L/F is a quadratic field extension, then
$$K_n(L) = r_{L/F}(K_{n-1}(F)) \cdot K_1(L)$$
for $n = 1, 2$.

- **Norm homomorphism:** Let L/F be a finite field extension, the standard norm homomorphism $L^* \to F^*$ can be viewed as a homomorphism $K_1(L) \to K_1(F)$.

  Suppose that the extension L/F is simple, we identify L with the residue field $F(y)$ for a closed point y. Let $\alpha \in K_2(L) = K_2(F(y))$, then there is a $\beta \in K_3(F(\mathbb{P}^1)$ satisfying $\partial_x(\beta) = \alpha$ if $x = y$ and $\partial_x(\beta) = 0$ otherwise. Let v be the discrete valuation of the field $F(\mathbb{P}^1)$ associated with the infinite point of the projective line. Set $c_{L/F}(\alpha) = \partial_v(\beta)$.

  In the general case, choose a sequence of simple field extensions
$$F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = L$$
and define
$$c_{L/F} = c_{F_1/F_0} \circ c_{F_2/F_1} \circ \cdots \circ c_{F_n/F_{n-1}}$$

- **Hilbert Theorem 90 for K_2.** Let L/F be a Galois quadratic extension and σ the generator of $\mathrm{Gal}(L/F)$. Then the sequence

$$K_2(L) \xrightarrow{1-\sigma} K_2(L) \xrightarrow{c_{L/F}} K_2(F)$$

is exact.

- Let L/F a quadratic extension. Then the sequence

$$k_2(F) \xrightarrow{r_{L/F}} k_2(L) \xrightarrow{c_{L/F}} k_2(F)$$

is exact.

Consider $u \in K_2(L)$ satisfying $c_{L/F}(u) = 2v$ for some $v \in K_1(F)$. Then

$$c_{L/F}(u - v_L) = 2v - 2v = 0$$

and by *Hilbert Theorem* 90, we have that $u - v_L = (1-\sigma)w$ for some $w \in K_2(L)$. Hence,

$$u = v_L + (1-\sigma)w = (v + c_{L/F}(w))_L - 2\sigma w$$

- Let p be a prime integer, a field F is called p-special if the degree of every finite field extension of F is a power of p. A p-special field has the following property

Let L/F be a finite field extension, then there is a tower of field extensions

$$F = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = L$$

satisfying $[F_{i+1} : F_i] = p$ for all $i = 0, \ldots, n-1$.

- β is injective.

Consider $u \in K_2(F)$ a sum of n symbols, such that $\beta(u + 2K_2(F)) = 1$. We are going to show by induction that $u \in 2K_2(F)$.

For $n = 1$, $u = [a, b]$, since $\beta[a, b] = (\frac{a,b}{F})$ is isotropic, there is $x, y \in F$ such that $b = x^2 - ay^2$, so from 6.3 $[a, b] \in K_2(F)$.

For $n = 2$, $u = [a, b] + [c, d]$, since $\beta(u + 2K_2(F)) = 1$, then $(\frac{a,b}{F}) \otimes (\frac{c,d}{F})$ is split, which is equivalent to $(\frac{a,b}{F}) \cong (\frac{c,d}{F})$. By Corollary 4.2 we may assume that $a = c$ and hence $u = [a, bd]$, so the statement follows from the case $n = 1$.

Suppose now that $u = [a, b] + v$ where $v \in K_2(F)$ is the sum of $n-1$ symbols. We may assume that $[a, b] \notin 2K_2(F)$, so the algebra $Q = (\frac{a,b}{F})$ is not split. Let

C be the corresponding conic curve and $L = F(C)$. The conic is given by the equation

$$at_1^2 + bt_2^2 = abt_3^2$$

Set $x = t_1/t_3$ and $y = t_2/t_3$, so the above equation becomes

$$b^{-1}x^2 + a^{-1}y^2 = 1$$

Thus in $K_2(L)$ is true

$$0 = [b^{-1}x^2, a^{-1}y^2] = 2[x, a^{-1}y^2] - 2[b, y] - [a, b]$$

Therefore $[a.b] = 2r$ in $K_2(L)$ with $r = [x, a^{-1}y^2] - [b, y]$. Since the quaternion algebra $(\frac{a,b}{F})$ is split over L, we have that $\beta_L(v_L + 2K_2(L)) = 1$.

By induction hypothesis, $v_L = 2w$ for some $w \in K_2(L)$. Set $c_x = \partial_x(w)$ for every $x \in C$. Since

$$c_x^2 = \partial_x(2w) = \partial_x(v_L) = 1$$

we have $c_x = (-1)^{n_x}$ for $n_x = 0$ or 1, by the previous remarks, the degree of every point of C is even so

$$\sum_{x \in C} n_x \deg(x) = 2m$$

for some $m \in \mathbb{Z}$. As every degree zero divisor on C is principal, there is a function $f \in L^*$ with

$$\operatorname{div}(f) = \sum n_x x - mp$$

where p is a non trivial residue of the element r and $\partial_r(p) = -1$.

Set $w' = w + [1, f] + kr \in K_2(L)$, where $k = m + n_p$. If $x \in C$ is a point different from p, then

$$\partial_x(w') = \partial_p(w)(-1)^{n_x} = 1$$

and

$$\partial_p(w') = \partial_p(w)(-1)^m(-1)^k = (-1)^{n_p+m+k} = 1$$

By the above exact sequence, we have that $w' = s_L$ for some $s \in K_2(F)$. Thus

$$v_L = 2w = 2w' - 2kr = 2s_L - [a^k, b]_L$$

Set $v' = v - 2s + [a^k, b] \in K_2(F)$; we have that $v'_L = 0$. The conic C has a rational point over the quadratic extension $E = F(\sqrt{a})$. Since the field extension $E(C)/E$ is purely transcendental and $v'_{E(C)}$ and therefore $2v' = 0$.

Since $v' = 0 \in k_2(F)$, there is a $d \in F^*$ such that $v = [-1, d]$, hence v is the sum of two symbols $[a^k, b]$ and $[-1, d]$ modulo $2K_2(F)$. Therefore, we are reduced to the case $n = 2$ that has already been considered.

- β is surjective.

Let $s \in {}_2Br(F)$, by induction on the index of s, we prove that $s \in \mathrm{Im}(\beta)$. First suppose that the field F is 2-special, by the previous remark, there is a quadratic extension L/F with $\mathrm{ind}(s_L) < \mathrm{ind}(s)$. By induction hypothesis, $s_L = \beta_L(u)$ for some $u \in k_2(L)$. It is easy to notice that $\beta \circ c_{L/F} = c_{L/F} \circ \beta_L$ and therefore

$$\beta(c_{L/F}(u)) = c_{L/F}(\beta_L(u)) = c_{L/F}(s_L) = 1$$

Since we already proved that β is injective, it follows that $c_{L/F}(u) = 0$. So by the above exact sequence, we have that $u = v_L$ for some $v \in k_2(F)$. Therefore,

$$\beta(v)_L = \beta_L(v_L) = \beta_L(u) = s_L$$

hence, $s - \beta(v)$ splits over L and must be the class of a quaternion algebra. Consequently, $s - \beta(v) = \beta(w)$ for some $w \in k_2(F)$. So $s = \beta(v + w) \in \mathrm{Im}(\beta)$.

In the general case, apply the proof to a maximal odd degree extension of F, and there exist an odd degree extension E/F such that $s_E = \beta_E(v)$ for some $v \in k_2(E)$. Finally we obtain

$$s = c_{E/F}(s_E) = c_{E/F}(\beta_E(v)) = \beta(c_{E/F}(v))$$

□

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