Conjugation Spaces
Comprehensive Examination Part II

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1 Introduction

Let X be a topological space, a continuous action of a topological group G over X allow to study properties and invariants throughout symmetries of X; we want to inherit to X an algebraic object that reflects both the topology and the group action since the usual cohomology ring $H^*(X)$ does not consider the action. The ring $H^*(X/G)$ is not a suitable candidate since in the orbit space X/G some pathologies may appear if the action of G on X is not free. We might consider then the ring $H^*(\widetilde{X}/G)$ where $\widetilde{X}$ is some particular topological space which is homotopy equivalent to X and G acts on it freely.

We are particularly interested in involutions of X, namely, a continuous map $\tau : X \to X$ such that $\tau^2 = id$ induces an action of the group $G = \{id, \tau\}$ on X. Moreover, if X is a complex manifold and τ is the complex conjugation, some algebraic relations between the rings $H^*(X)$ and $H^*(\widetilde{X}/G)$ occur. This notion can be generalized under suitable conditions over the cohomology rings leading to the concept of **Conjugation Space**.

In section §2. we present the generalities of the Borel construction $\widetilde{X}/G$ and the G-equivariant cohomology ring $H^*(\widetilde{X}/G)$, the proofs of that section are not presented and can be found in [3]. In section §3. is developed the theory of Conjugation Spaces, main properties and examples are presented. Throughout all the document, $H^*(X)$ will denote the singular cohomology ring $H^*(X;\mathbb{Z}_2)$ where $\mathbb{Z}_2$ denotes the field of two elements. For basic and introductory results on Algebraic Topology we will refer [1].

2 Spaces With Involution

Let X be a topological space. An **involution** on X is a continuous map $\tau : X \to X$ such that $\tau^2 = id$. If we denote by $G = \{\tau, id\}$ the cyclic group of order 2, an involution on X is equivalent to X being a G-space, that is, a continuous action of G over X.

The **fixed point subspace of** X, X^G is defined by

$$X^G = \{x \in X : \tau(x) = x\}$$

So the complement of X^G in X is the subspace where the action of G is free.

Given two G-spaces X and Y, a G**-map** $f : X \to Y$ is a continuous function that commutes with the involutions, namely $f \circ \tau_X = \tau_Y \circ f$. Let $f^G : Y^G \to X^G$ denote the restriction of f to the fixed point subspaces. Two G-maps f_1, f_2 are G-homotopic if there exist a homotopy $F : X \times I \to Y$ connecting them such that for any $x \in X$ and $t \in I$, $\tau_Y F(x,t) = F(\tau_X(x), t)$.

Remark 2.1. Suppose that X is a G-space which has a CW-complex structure such that for each n, there is an action of G on the set of n-cells Λ_n, and a G-characteristic map $\psi_n : \Lambda_n \times D^n \to X$. (Here the action of G on $\Lambda_n \times D$ is given by $\tau(\lambda, x) = (\tau(\lambda), x)$). In this case we say that X is a G**-**CW**-complex**.

Observe that if X is a G-CW-complex then the quotient space X/G inherits a CW-structure, with the set of n-cells equal to Λ_n/G.

Example 2.2. The sphere S^n can be obtained from S^{n-1} by adjunction of two n-cells D^n attached by the identity map on the boundary S^{n-1}. Starting from $S^0 = \{\pm 1\}$ we have a CW-structure on S^n with two k-cells and whose k-skeleton is S^k, for $k \leq n$. This is also a G-CW-structure over S^n for the involution given by the antipodal map $x \mapsto -x$. The quotient space S^n/G is $\mathbb{R}P^n$ which inherits a CW-structure. This construction also is applied for the inductive limits S^∞ and $\mathbb{R}P^\infty$.

Definition 2.3. Let X be a space with involution τ. **The Borel construction** X_G is the quotient space

$$X_G = S^\infty \times_G X = (S^\infty \times X)/\sim \tag{2.3.1}$$

where $\sim$ is the equivalence relation $(z, x) \sim (-z, \tau(x))$.

If $f : X \to Y$ is a G-map, then the map $id \times f : S^\infty \times X \to S^\infty \times Y$ induces a continuous map $f_G : X_G \to Y_G$. Also, if X and Y have the same G-homotopy type, then X_G and Y_G have the same homotopy type.

Example 2.4. Consider the constant map $X \to pt$, since $pt_G = \mathbb{R}P^\infty$, the induced map $p : X_G \to \mathbb{R}P^\infty$ is given by $p([z,x]) = \hat{p}(z)$, where $\hat{p} : S^\infty \to \mathbb{R}P^\infty$ is the 2-fold covering projection.

Example 2.5. Suppose that the involution on X is trivial, that is $\tau(x) = x$ for all $x \in X$. Then we have a homeomorphism $X_G \xrightarrow{\approx} \mathbb{R}P^\infty \times X$, induced by the continuous maps $p : X_G \to \mathbb{R}P^\infty$ (from Example 2.4) and $X_G \to X$ (induced by the projection $S^\infty \times X \to X$).

Lemma 2.6. *Let X be a space with involution τ.*

1. *The map $p : X_G \to \mathbb{R}P^\infty$ is a locally trivial fiber bundle with fiber homeomorphic to X.*

2. *if τ has a fixed point, then p admits a section. More precisely, each $v \in X^G$ provides a section $s_v : \mathbb{R}P^\infty \to X_G$ of p.*

3. *The quotient map $S^\infty \times X \to X_G$ is a 2-fold covering.*

4. *If X is a free G-space and Hausdorff, then morphism $q^* : H^*(X/G) \to H^*(X_G)$ (induced by the projection G-map $\widetilde{q} : S^\infty \times X \to X$) is an isomorphism of graded algebras. Furhtermore, if X is a G-CW-complex, the map q is a homotopy equivalence.*

Moreover, we have the following consequence:

Corollary 2.7. *Let X be a finite dimensional G-CW-complex. Then X has a fixed point if and only if the homomorphism $p^* : H^*(\mathbb{R}P^\infty) \to H^*(X_G)$ is injective.*

Definition 2.8. Let X be a space with involution τ. The **G-equivariant cohomology** $H^*_G(X)$ is the cohomology algebra defined by

$$H^*_G(X) = H^*(X_G). \tag{2.8.1}$$

Recall that $H^*(\mathbb{R}P^\infty) \approx \mathbb{Z}_2[u]$ is the polynomial ring where u is a variable in degree 1. (See [1, Theorem 3.19]). In particular, $H^*_G(pt) = \mathbb{Z}_2[u]$. Observe that for any G-space, the homomorphism $p^* : H^*(\mathbb{R}P^*) \to H^*_G(X)$ gives $H^*_G(X)$ the structure of a $\mathbb{Z}_2[u]$-algebra

Example 2.9. Suppose that the action of G on X is trivial, that is, $X^G = X$. From Example 2.5 we have that $X_G = \mathbb{R}P^\infty \times X$, so by the Künneth formula

$$H^*_G(X) \approx \mathbb{Z}_2[u] \otimes H^*(X) \approx H^*(X)[u] \tag{2.9.1}$$

Definition 2.10. Let X be a G-space, for any point $z \in S^\infty$, we have a map $i_z : X \to X_G$ given by $i_z(x) = [z, x]$. Since S^∞ is path-connected, we have a well defined homomorphism (independent of z)

$$\rho = i_z^* : H^*_G(X) \to H^*(X) \tag{2.10.1}$$

We call ρ the **forgetful homomorphism**, and we say that X is **equivariantly formal** if ρ is surjective.

Observe that ρ is functorial, that is, if $f : Y \to X$ is a G-map, the following diagram

$$\begin{tikzcd} H^*_G(X) \arrow[r, "\rho_X"] \arrow[d, "f^*_G"'] & H^*(X) \arrow[d, "f^*"] \\ H^*_G(Y) \arrow[r, "\rho_Y"'] & H^*(Y) \end{tikzcd}$$

is commutative.

Consider the subalgebra of $H^*(X)$ given by

$$H^*(X)^G = \{a \in H^*(X) : \tau^*(a) = a\}. \tag{2.10.2}$$

Let $z \in S^\infty$, for any $b \in H^*_G(X)$, we have that

$$\tau^* \circ \rho(b) = \tau^* \circ i^*_z(b) = (i_z \circ \tau)^*(b) = i^*_{-z}(b) = \rho(b).$$

This shows that $\rho(H^*_G(X)) \subseteq H^*(X)^G$.

Proposition 2.11. *Let X be a G-space with $X^G \neq \emptyset$. Suppose that $\widetilde{H}^i(X) = 0$ for $0 \leq i < r$. Then there is a short exact sequence:*

$$0 \to H^r(\mathbb{R}P^\infty) \xrightarrow{p^*} H^r_G(X) \xrightarrow{\rho} H^r(X)^G \to 0. \tag{2.11.1}$$

Remark 2.12. If X is equivariantly formal, for each $k \in \mathbb{N}$, we can choose a $\mathbb{Z}_2$-linear map $\sigma : H^k(X) \to H^k_G(X)$ such that $\rho\sigma = id$. This gives rise to an additive section $\sigma : H^*(X) \to H^*_G(X)$ of ρ . The Leray–Hirsch Theorem ([3, Theorem 4.1.17]) applied to the locally trivial fiber bundle $X \to X_G \to \mathbb{R}P^\infty$ implies that the induced map $\widetilde{\sigma} : H^*(X)[u] \to H^*_G(X)$is an isomorphism of $\mathbb{Z}_2[u]$-modules (but not in general an isomorphism of rings). Therefore, for an equivariantly formal space, $\ker(\rho)$ is the ideal generated by u.

Proposition 2.13. *Consider the ideal of $H^*_G(X)$*

$$Ann(u) = \{x \in H^*_G(X) : ux = 0\}.$$

The following conditions are equivalent:

1. *X is equivariantly formal.*
2. *$H^*_G(X)$ is a free $\mathbb{Z}_2[u]$-module.*
3. *$Ann(u) = 0$.*

Let $r : H^*_G(X) \to H^*_G(X^G)$ be the homomorphism induced by the inclusion $X^G \hookrightarrow X$. As a consequence of Proposition 2.13 we have the following result:

Corollary 2.14. *Let X be a G-space and suppose that $r : H^*_G(X) \to H^*_G(X^G)$ is injective. Then X is equivariantly formal.*

The converse of the above result is not true in general; for instance, consider $X = S^\infty$ together with the antipodal action. Since S^∞ is contractible, $H^*(S^\infty) \approx H^*(pt) = \mathbb{Z}_2$; moreover, by Lemma 2.6, $H^*_G(S^\infty) \approx H^*(S^\infty/G) = H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[u]$. Therefore, the map $\rho : H^*_G(X) \to H^*(X)$ is surjective and thus X is equivariantly formal. However, $H^*_G(X^G) = H^*_G(\emptyset) = 0$ implies that the map $r : H^*_G(X) \to H^*_G(X^G)$ is not injective.

Definition 2.15. For a G-space X, let

$$h^*_G(X) = \mathbb{Z}_2[u, u^{-1}] \otimes H^*_G(X) \tag{2.15.1}$$

denote the **localization** of the cohomology algebra $H^*_G(X, Y)$.

Recall that $\mathbb{Z}_2[u, u^{-1}]$ is $\mathbb{Z}$-graded with $\mathbb{Z}_2[u, u^{-1}]^k = \text{span}_{\mathbb{Z}_2}(u^k)$. Therefore, $h^*_G(X, Y)$ is a $\mathbb{Z}$-graded $\mathbb{Z}_2[u, u^{-1}]$-algebra with

$$h^*_G(X, Y)^k = \bigoplus_{i+j=k} \text{span}_{\mathbb{Z}_2}(u^i) \otimes H^j_G(X, Y)/\sim$$

where $\sim$ is the equivalence relation generated by $u^{i+1} \otimes a \sim u^i \otimes ua$.

Example 2.16. Suppose that X is a free finite dimensional G-CW-complex. By Lemma 2.6, $H^*_G(X) \approx H^*(X/G)$. Since X/G is a finite dimensional CW-complex, $H^*_G(X)$ is a torsion $\mathbb{Z}_2[u]$-module. Therefore, the localization $h^*_G(X) = 0$. On the other hand, since the action is free, $X^G = \emptyset$ and thus $h^*_G(X^G) = 0$ as well. So in this case there is a trivial isomorphism $h^*_G(X) \approx h^*_G(X^G)$.

The isomorphism presented in the previous example can be generalized under suitable conditions, as the following theorem states.

Theorem 2.17 (Localization Theorem)**.** *Let X be a finite dimensional G-CW-complex. Then the inclusion $X^G \hookrightarrow X$ induces an isomorphism*

$$h^*_G(X) \xrightarrow{\approx} h^*_G(X^G)$$

of $\mathbb{Z}$-graded $\mathbb{Z}_2[u, u^{-1}]$-algebras.

The finite dimensional hypothesis is necessary in this theorem; for instance, if $X = S^\infty$ with the antipodal involution, $X^G = \emptyset$, but $h^*_G(X) = \mathbb{Z}_2[u, u^{-1}]$ since $H^*_G(X) = \mathbb{Z}_2[u]$.

As a consequence of the Localization Theorem we have

Corollary 2.18. *Let X be a G-space and suppose that X is equivariantly formal. Then the restriction homomorphism $r : H^*_G(X) \to H^*_G(X^G)$ is injective.*

3 Conjugation Spaces

There are examples of G-spaces for which there exists a ring isomorphism $\kappa : H^{2*}(X) \to H^*(X^G)$, where $H^{2*}(X)$ denotes the subalgebra of even degree elements in $H^*(X)$. This isomorphism is part of an interesting structure on equivariant cohomology, which generalizes these examples into the concept of a *Conjugation Space*. In this section we develop this concept and present properties and examples for these spaces.

Definition 3.1. Let X be a G-space, $\rho : H^{2*}_G(X) \to H^{2*}(X)$ the forgetful homomorphism and $r : H^*_G(X) \to H^*_G(X^G)$ the restriction homomorphism. A **cohomology frame** for X is a pair (κ, σ) satisfying

- $\kappa : H^{2*}(X) \to H^*(X^G)$ is an additive isomorphism dividing the degree in half.
- $\sigma : H^{2*}(X) \to H^{2*}_G(X)$ is an additive section of ρ.

- κ, σ satisfy the **conjugation equation** in the ring $H^*_G(X^G)$; namely, for any $a \in H^{2m}(X)$, $m \in \mathbb{N}$,
$$r \circ \sigma(a) = \kappa(a)u^m + p_m \tag{3.1.1}$$
where p_m denotes *some* polynomial in $H^*_G(X^G) \approx H^*(X^G)[u]$ of degree less than m.

An involution τ admitting a cohomology frame is called a **conjugation**. Additionally, if $H^{odd}(X) = 0$ then X together with an involution is a **conjugation space**.

Notice that the existence of σ is equivalent to ρ being surjective and thus X is *equivariantly formal* (see Definition 1.9). We have the following examples of Conjugation Spaces.

Example 3.2. Let $D = D^{2n} = \{x \in \mathbb{R}^{2n} : \|x\| \leq 1\}$ and τ the involution given by $\tau(x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}) = (-x_1, \dots, -x_n, x_{n+1}, \dots, x_{2n})$. We have that D^G is homeomorphic to a disc of dimension n and therefore there is a canonical ring isomorphism $\kappa : H^{2*}(D) \to H^*(D^G)$. On the other hand, since D is G-homotopic to a point, $D_G \approx \mathbb{R}P^\infty$. Then the map $\rho : H^*_G(D) \to H^*(D)$ coincides with the evaluation at $u = 0$ which is clearly surjective. Let σ be an additive section of ρ; for any $a \in H^{2m}(D)$, the conjugation equation holds trivially if $m > 0$. In the case $m = 0$, the equation $r \circ \sigma(a) = \kappa(a)$ follows immediately by definitions of the maps involved.

Example 3.3. Let D as in Example 3.2 and set $\Sigma = D/\partial D$. The action of G over D induces a well defined action over Σ. Since Σ is homeomorphic to the sphere S^{2n} and Σ^G to the sphere S^n, there is a obvious isomorphism $\kappa : H^{2*}(\Sigma) \to H^*(\Sigma^G)$, namely, sending the non-trivial element $a \in H^{2n}(\Sigma)$ to the non trivial element $b \in H^n(\Sigma^G)$.
By Proposition 2.11 and as $H^{2n}(\Sigma)^G = H^{2n}(\Sigma)$, the map $\rho : H^{2n}_G(\Sigma) \to H^{2n}(\Sigma)$ is surjective; in particular, this implies that Σ is equivariantly formal. Let σ be a section of ρ.
We only need to check that the conjugation equation holds for $a \in H^{2n}(\Sigma)$. As in Example 2.9, we have that $H^{2n}_G(\Sigma^G) \approx (H^*(\Sigma^G)[u])^{2n}$, then
$$r(\sigma(a)) = \kappa(a)u^n \tag{3.3.1}$$
follows from r being injective (Corollary 2.14).

Remark 3.4. Let X be a conjugation space. On $H^0(X)$, the map κ coincides with the restriction homomorphism $\widetilde{r} : H^0(X) \to H^0(X^G)$. Indeed, for any $a \in H^0(X)$, we have that $\kappa(a) = r \circ \sigma(a)$ using the conjugation equation; on the other hand, there is a commutative diagram

$$\begin{tikzcd}
H^0_G(X) \arrow[d, "r"'] & H^0(X) \arrow[l, "\sigma=\rho^{-1}"'] \arrow[dl, "\kappa"'] \arrow[d, "\widetilde{r}"] \\
H^0_G(X^G) \arrow[r, "\approx"] & H^0(X^G)
\end{tikzcd}$$

Thus we have $\kappa(a) = r \circ \sigma(a) = \widetilde{r}(a)$.

In particular, the last remark implies that κ and σ are morphisms that preserve the units of the rings.

Theorem 3.5 (Multiplicativity Theorem)**.** *Let X be a conjugation space, then κ and σ are ring homomorphisms.*

Proof. First we prove that $\sigma(ab) = \sigma(a)\sigma(b)$ for all $a \in H^{2k}(X)$, $b \in H^{2l}(X)$ Let $m = k + l$.

Since $\rho : H^0_G(X) \to H^0(X)$ is an isomorphism, the claim follows for $m = 0$. So we may suppose that $m > 0$. Since σ is a section of ρ, which is a ring homomorphism, we have

$$\rho(\sigma(ab)) = ab, \quad \text{and} \quad \rho(\sigma(a)\sigma(b)) = \rho(\sigma(a))\rho(\sigma(b)) = ab \tag{3.5.1}$$

Thus, $\sigma(ab) = \sigma(a)\sigma(b) \bmod \ker \rho$. Recall that $H^{odd}(X) = 0$, and by Remark 2.12 we have a $\mathbb{Z}[u]$-module isomorphism $H^*_G(X) \approx H^*(X)[u]$ and $\ker(\rho) = \langle u \rangle$; therefore it follows,

$$\sigma(ab) = \sigma(a)\sigma(b) + \sigma(d_{2m-2})u^2 + \cdots + \sigma(d_0)u^{2m} \tag{3.5.2}$$

where $d_i \in H^i(X)$. We want to show that $d_{2m-2} = \cdots = d_0 = 0$.

Applying the restriction homomorphism r to the previous equation, we obtain

$$\begin{aligned} \kappa(ab)u^m + \widetilde{p_m} &= k(a)k(b)u^m + p_m + (\kappa(d_{2m-2})u^{m-1} + p_{m-1})u^2 + \cdots + \kappa(d_0)u^{2m} \\ &= \kappa(d_0)u^{2m} + p_{2m} \end{aligned}$$

By comparing the $2m$-th terms, we get $\kappa(d_0) = 0$, and thus $d_0 = 0$ by injectivity of κ. So we can rewrite the equation (3.5.2) as

$$\kappa(ab)u^m + \widetilde{p_m} = \kappa(d_2)u^{2m-1} + p_{2m-1} \tag{3.5.3}$$

obtaining $d_2 = 0$ as before. Following the process inductively, we get $d_{2m-2} = \cdots = d_2 = d_0 = 0$, proving that σ is a ring homomorphism.

To prove that $\kappa(ab) = \kappa(a)\kappa(b)$, using the conjugation equation (3.1.1) on both sides of

$$r\sigma(ab) = r\sigma(a)r\sigma(b),$$

we have

$$\begin{aligned} \kappa(ab)u^m + p_m &= (\kappa(a)u^k + p_k)(\kappa(b)u^l + p_l) \\ &= \kappa(a)\kappa(b)u^m + \widetilde{p_m}. \end{aligned}$$

Therefore, $\kappa(ab) = \kappa(a)\kappa(b)$ by comparing degrees. □

As an immediate corollary, following Remark 2.12, we have

Corollary 3.6. *The map* $\widetilde{\sigma} : H^*(X)[u] \to H^*_G(X)$ *induced by* σ*, is an isomorphism of* $\mathbb{Z}_2[u]$*-algebras.* □

Corollary 3.7. *Let* X *be a conjugation space, then the restriction homomorphism* $r : H^*_G(X) \to H^*_G(X^G)$ *is injective.*

Proof. Let $x \in \ker(r)$. Since there is an isomorphism $H^*_G(X) \approx H^*(X)[u]$ induced by σ (Corollary 3.6), write

$$x = \sigma(y)u^k + p_k \in H^{2n+k}_G(X)$$

for some $y \in H^{2n}(X)$; without loss of generality, suppose that k is minimal. From the conjugation equation we get

$$0 = r(x) = r(\sigma(y)u^k + p_m) = \kappa(y)u^{n+k} + p_{n+k}; \tag{3.7.1}$$

therefore $y = 0$ since κ is an isomorphism, and thus $x = 0$. □

Now we focus on the naturality of the cohomology frames, which follows in part form the naturality given by the following diagrams.

Lemma 3.8. *Let* X *and* Y *be conjugation spaces,*

1. *Let* $f : Y \to X$ *be a* G*-map, and* $f^G : Y^G \to X^G$ *the restriction to the fixed point subspaces. Then the following diagram*

$$\begin{array}{ccc} H^*(X^G)[u] & \xrightarrow{(f^G)^*[u]} & H^*(Y^G)[u] \\ \downarrow \approx & & \downarrow \approx \\ H^*_G(X^G) & \xrightarrow{(f^G_G)^*} & H^*_G(Y^G) \end{array}$$

   is commutative, where $(f^G)^*[u]$ *is the polynomial extension of the map* $(f^G)^*$.

2. *Let* r_X *and* r_Y *be the restriction homomorphism for* X *and* Y *respectively. Then the following diagram*

$$\begin{array}{ccc} H^*_G(X) & \xrightarrow{r_X} & H^*_G(X^G) \\ \downarrow (f_G)^* & & \downarrow (f^G_G)^* \\ H^*_G(Y) & \xrightarrow{r_Y} & H^*_G(Y^G) \end{array}$$

   is commutative.

□

Now we can show the following result:

Proposition 3.9 (Naturality of Cohomology Frames)**.** *Let X and Y be conjugation spaces with (κ_X, σ_X) and (κ_Y, σ_Y) the respective cohomology frames. The conjugation space structure is natural, that is, the following diagrams are commutative:*

$$(1)\quad \begin{tikzcd} H^{2*}(X) \arrow[r, "\sigma_X"] \arrow[d, "f^*"'] & H_G^{2*}(X) \arrow[d, "f_G^*"] \\ H^{2*}(Y) \arrow[r, "\sigma_Y"'] & H_G^{2*}(Y) \end{tikzcd}$$

$$(2)\quad \begin{tikzcd} H^{2*}(X) \arrow[r, "\kappa_X"] \arrow[d, "f^*"'] & H^*(X^G) \arrow[d, "(f^G)^*"] \\ H^{2*}(Y) \arrow[r, "\kappa_Y"'] & H^*(Y^G) \end{tikzcd}.$$

Proof. Fix k a natural number and let ρ_X and ρ_Y be the respective forgetful homomorphism. Since $f^* \circ \rho_X = \rho_Y \circ f_G^*$, for any $a \in H^{2k}(X)$ it follows,

$$\rho_Y \circ f_G^* \circ \sigma_X(a) = f^* \circ \rho_X \circ \sigma_X(a) = f^*(a) = \rho_Y \circ \sigma_Y \circ f^*(a) \tag{3.9.1}$$

From the above equation, we get the commutativity of the diagram (1) modulo ker ρ_Y. As in the proof of Theorem 3.5, write

$$f_G^* \sigma_X(a) = \sigma_Y f^*(a) + \sigma_Y(d_{2k-2})u^2 + \cdots + \sigma_Y(d_0)u^{2k} \tag{3.9.2}$$

for some $d_i \in H^i(Y)$. Applying r_Y to both sides of this equation, by Lemma 3.8 we get on the left hand side

$$\begin{aligned} r_Y(f_G^* \sigma_X(a)) &= (f_G^G)^*(r_X \sigma_X(a)) \\ &= (f_G^G)^*(\kappa_X(a)u^k + p_k) \\ &= (f_G^G)^*(\kappa_X(a))u^k + \widetilde{p_k} \end{aligned}$$

and for the right hand side we obtain,

$$r_Y(\sigma_Y f^*(a) + \sigma_Y(d_{2k-2})u^2 + \cdots + \sigma_Y(d_0)u^{2k}) = \kappa_Y(d_0)u^{2k} + p_{2k}$$

Combining both sides we conclude that $\kappa_Y(d_0) = 0$ and then $d_0 = 0$ because of the injectivity of κ_Y. Therefore, the equation (3.8.2) can be rewritten as

$$r_Y(f_G^* \sigma_X(a)) = \kappa_Y(d_2)u^{2k-2} + p_{2k-2} \tag{3.9.3}$$

and as before, it follows $d_2 = 0$. We can continue this process to finally get

$$f_G^* \circ \sigma_X(a) = \sigma_Y \circ f^*(a). \tag{3.9.4}$$

To prove the commutativity of the diagram (2), apply r_Y to both sides of equation (3.8.4), and the conjugation equation and Lemma 3.8 imply

$$(f^G)^*(\kappa_X(a))u^k + p_k = \kappa_Y(f^*(a))u^k + \widetilde{p_k}; \tag{3.9.5}$$

therefore, comparing the leading term u^k and using that $(f_G^G)^* = (f^G)^*[u]$ we obtain

$$(f^G)^* \circ \kappa_X(a) = \kappa_Y \circ f^*(a) \tag{3.9.6}$$

□

Corollary 3.10 (Uniqueness of Cohomology Frames)**.** *Let* (κ, σ) *and* (κ', σ') *cohomology frames for an involution* τ *on* X*. Then* $(\kappa, \sigma) = (\kappa', \sigma')$*.*

Proof. Set $Y = X$ and $f = id$ on proposition 3.9, this gives $\kappa = \kappa'$ and $\sigma = \sigma'$. □

Recall that on a conjugation space X, σ induces an isomorphism

$$\widetilde{\sigma} : H^*(X)[u] \to H^*_G(X)$$

(see Corollary 2.4). In fact, proposition 3.9 shows that this isomorphism is natural and gives the following result analogous to Lemma 2.5.

Corollary 3.11. *For any G-map $f : Y \to X$ between conjugation spaces, the diagram*

$$\begin{tikzcd}
H^*(X)[u] \arrow[r, "\widetilde{\sigma}_x", "\approx"'] \arrow[d, "f^*[u]"'] & H^*_G(X) \arrow[d, "f^*_G"] \\
H^*(Y)[u] \arrow[r, "\widetilde{\sigma}_y", "\approx"'] & H^*_G(Y)
\end{tikzcd}$$

is commutative, where $f^*[u]$ is the polynomial extension of the map f^*.

Many examples of conjugation spaces are constructed by successively attaching of cells homeomorphic to a disc in $\mathbb{C}^n$, together with the complex conjugation. This construction is analogous to the standard construction for CW-complexes, so in this section we focus on presenting some tools for construction of conjugation complexes.

The examples 3.2 and 3.3 motivate the following definition:

Definition 3.12. A **conjugation cell of dimension** $2n$ is the closed unit disk D in $\mathbb{R}^{2n}$ with a linear involution τ which has exactly n eigenvalues equal to -1. Notice that D can be seen as the disc in $\mathbb{C}^n$ and τ the complex conjugation. The space $\Sigma = D/\partial D$ is called a **conjugation sphere of dimension** $2n$**.**

Let Y be a topological space with an involution τ and D be a conjugation cell of dimension $2k$. Let $\alpha : S \to Y$ be a G-map where S denotes the boundary of D. Then the involutions on Y and D induces an involution on the quotient space

$$X = Y \cup_\alpha D = Y \amalg D/\{y = \alpha(u) : u \in S\}. \tag{3.12.1}$$

We say that X is obtained from Y by attaching a conjugation cell of dimension $2k$. Observe that the fixed point subspace X^G is obtained from Y^G by attaching a k-cell (in the standard sense of CW-complexes).

More genereally, we can attach to Y a set Λ of $2k$-conjugation cells, using a G-map

$$\alpha : \coprod_{\lambda \in \Lambda} S_\lambda \to Y.$$

As before, we have an involution over the resulting space X and the fixed point set X^G is obtained from Y^G by attaching a collection of k-cells indexed by the same set Λ.

Remark 3.13. The notion of G-spaces, G-equivariant cohomology and conjugation spaces can be generalized to topological pairs (X, Y), where X is a G-space and Y is a G-stable subspace of X. Such a G-pair together with a cohomology frame (κ, σ) is called a Conjugation pair. The main fact about these conjugation pairs is that if (X, Y) is a Conjugation pair and Y is a Conjugation space, then X is a Conjugation space. (See Proposition 4.1 in [2].)

The main fact about the construction of attaching conjugation cells to a conjugation space is the following:

Proposition 3.14. *Let Y be a conjugation space and let X be obtained from Y by attaching a collection of $2k$-conjugation cells. Then X is a conjugation space.*

Proof. Suppose that X is obtained from Y by attaching exactly one $2n$-conjugation cell $D = D_\lambda$. According to Remark 3.13, We will show that (X, Y) is a conjugation pair. Using excision, we have $H^*(X, Y) \approx H^*(D, S)$ where $S = \partial D_\lambda$. As in example 3.3, one can show that there is a Cohomology frame $(\sigma_\lambda, \kappa_\lambda)$ such that the equation

$$r_\lambda(\sigma_\lambda(a)) = \kappa_\lambda(a) u^n \tag{3.14.1}$$

holds, where $a \in H^n(D, S)$ is the non-zero element.

For the general case, suppose that X is obtained from Y by attaching a set Λ of $2n$-conjugation cells. Set $D = \coprod_{\lambda \in \Lambda} D_\lambda$ and $S = \coprod_{\lambda \in \Lambda} \partial D_\lambda$. Then we have

$$H^*(D, S) = \prod_{\lambda \in \Lambda} H^*(D_\lambda, \partial D_\lambda) \tag{3.14.2}$$

$$H^*_G(D, S) = \prod_{\lambda \in \Lambda} H^*_G(D_\lambda, \partial D_\lambda). \tag{3.14.3}$$

Setting $\kappa = \prod_{\lambda \in \Lambda} \kappa_\lambda$, $\sigma = \prod_{\lambda \in \Lambda}$, it follows as in the previous case that

$$r(\sigma(a)) = \kappa(a) u^n$$

for any $a \in H^{2n}(X, Y) \approx H^{2n}(D, S)$ (by excision), where $r = \prod_{\lambda \in \Lambda} r_\lambda$. Therefore (X, Y) is a conjugation pair and so X is a conjugation space. □

To complete the construction of conjugation complexes, we need the following Lemma (for details see Proposition 4.6 in [2]).

Lemma 3.15 (Direct Limits)**.** *Let* (X_i, f_{ij}) *be a directed system of conjugation spaces and* G*-equivariant inclusions, indexed by a directed set* I*. Suppose that each space* X_i *is* T_1*. Then* $X = \varinjlim X_i$ *is a conjugation space.* □

Definition 3.16. A space X is a **conjugation complex** if it is equipped with a filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots X = \bigcup_{k=0}^{\infty} X_k \tag{3.16.1}$$

where X_k is obtained by attaching a collection of conjugation cells to X_{k-1} (indexed by a set $\Lambda_k(X)$. The topology on X is the direct limit topology.

By Proposition 3.14 and Lemma 3.15 it follows that a conjugation complex X is a conjugation space.

Example 3.17. The complex projective spaces $\mathbb{C}P^n$ and the complex Grassmannian Manifolds $Gr_k(\mathbb{C}^n)$ together with the complex conjugation, are conjugation complexes (and therefore conjugation spaces) for any $1 \leq n \leq \infty$ with the standard CW-complex decomposition.

References

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