Automorphism group of compact Riemann surfaces

Sergio Chaves

July 21, 2018

⊠In this report we will study $\mathrm{Aut}(X)$, the automorphism group of a compact Riemann surface X (CRS for short), The main purpose will be to prove the *Hurwitz automorphism theorem* which states that $\mathrm{Aut}(X)$ is finite and its order is bounded by $84(g-1)$ where $g \geq 2$ is the genus of the Riemann surface. We will show that this bound is sharp, and even there exists infinitely many values of g where the equality is attached. Also, we will discuss further properties of the automorphisms of a Riemann surface and some explicit examples of $\mathrm{Aut}(X)$.

The basic theory of Riemann surfaces is assumed and will be omitted; however, we will include a few fundamental results that will be useful for the purposes of this document.

1 Preliminaries

In this section we will follow [Forster, 1991] for a review of important results in the theory of Riemann surfaces.

Let $f : X \to Y$ be a n-sheeted covering between compact Riemann surfaces. If $x \in X$ is a ramification point, then locally $f(z) = z^k$ for some charts $U \subseteq X$, $V \subseteq Y$ around $x, f(x)$ respectively and $k > 1$, we write by $\nu_x = k$ the ramification index of f at x. Since each x is an isolated point and X is compact, we have that X admits only finitely many ramification points. Therefore, the following expression

$$\nu_f = \sum_{x \in X} \nu_x - 1$$

is well defined. The Riemann-Hurwitz theorem states

Theorem 1.1 (Riemann - Hurwitz Theorem)**.** *Let g_x and g_y be the genus of X and Y respectively. Then*

$$g_x - 1 = n(g_y - 1) + \frac{\nu_f}{2}$$

or equivalently,

$$\chi(X) = n\chi(Y) - \nu_f.$$

For any Riemann surface X, let $\mathcal{M}(X)$ denote the field of meromorphic maps on X. Recall that any meromorphic function has as many zeros as poles.

Let D a divisor on X, write by $L(D) = \{f \in \mathcal{M}(X) : (f) + D \geq 0\}$ and $l(D) = \dim L(D)$.

Theorem 1.2 (Riemann-Roch Theorem). *Let X be a compact Riemann surface of genus g and K be a canonical divisor of X. Then*

$$l(D) - l(K - D) = \deg(D) + 1 - g$$

An important consequence of the Riemann-Roch theorem is the existence of non-constant meromorphic functions.

Proposition 1.3. *Let X be a CRS of genus g and let $a \in X$. There exists $f \in \mathcal{M}(X)$ that has a pole of order at most $g+1$ at a and it is elsewhere holomorphic.*

Proof. Consider the divisor $D = (g+1)\cdot a$, the Riemann-Roch theorem implies that $l(D) \geq \deg(D) + 1 - g = 2$. Therefore, for $f \in L(D)$, f has its only pole at a (otherwise $(f) + D \ngeq 0$) and it is of order at most $g+1$. □

2 The automorphism group

Let X be a Riemann surface, the set $\mathrm{Aut}(X) = \{f : X \to X$ biholomorphic $\}$ is a group under composition and it is called the automorphism group of X. When X is simply connected, (not necessarily compact) this group has an specific presentation. From the Uniformization theorem, X is either isomorphic to the complex plane $\mathbb{C}$, the upper half space $\mathbb{H}$ or the projective space $\mathbb{P}^1$

Proposition 2.1. $\mathrm{Aut}(\mathbb{C}) = \{f(z) = az + b : a \neq 0\}$ *and* $\mathrm{Aut}(\mathbb{P}^1) = \{f(z) = \dfrac{az+b}{cz+d} : ad - bc \neq 0\} \cong SL(2;\mathbb{C})/\pm I = PSL(2,\mathbb{C})$.

Proof. Let us start with the case of $X = \mathbb{P}^1$ since any automorphism of $\mathbb{C}$ can be extended to an automorphism of $\mathbb{P}^1$ sending ∞ to ∞. Let $f \in \mathrm{Aut}(\mathbb{P}^1)$, then $f \in \mathcal{M}(\mathbb{P}^1)$ and thus f is a rational function $f(z) = p(z)/q(z)$. Assume that p and q has no common zeros. Since f is injective, p can only have exactly one zero (from the fundamental theorem of algebra) and thus p is a linear map; hence, f has only one pole and thus q is also linear. That is,

$$f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$. The existence of f^{-1} is equivalent to $ad - bc \neq 0$. Moreover, dividing by $\pm\sqrt{ad-bc}$ we may assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})/\{\pm I\} = PSL(2,\mathbb{C})$ and so $\mathrm{Aut}(\mathbb{P}^1) \cong PSL(2,\mathbb{C})$. Now, any automorphism $f : \mathbb{C} \to \mathbb{C}$ can be extended to an automorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ by $f(\infty) = \infty$. If $f(z) = az + b/cz + d$ then $c = 0$ and $d \neq 0$ (otherwise $f(\infty) \neq \infty$) and so $f(z)$ is a linear map.

□

Proposition 2.2. $\mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R})$.

Proof. Let $f_A(z) = \frac{az+b}{cz+d}$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$. Now it is straightforward to check that $im(f_A(z)) = \frac{ad-bc}{|cz+d|^2} im(z)$ and so f_A is a well defined map $f : \mathbb{H} \to \mathbb{H}$. We will show that any $f \in \mathrm{Aut}(X)$ is of the form $f(z) = \frac{az+b}{cz+d}$. Consider the map $G : \mathbb{C} \to \mathbb{C}$ given by $G(z) = \frac{i-z}{i+z}$, this map induces a biholomorphism $G:\mathbb{H}\to D$ where D denotes the unit disc, and satisfying $G(i)=0$. Suppose that $f\in \mathrm{Aut}(X)$ is such that $f(i)=i$ and consider the map $g=G\circ f\circ G^{-1}:D\to D$, notice that $g(0)=0$ and thus from the Schwartz $g(z)=\lambda z$ with $\lambda=e^{i\theta}$ for some $\theta\in\mathbb{R}$. Now consider the matrix $A=\begin{pmatrix}\cos(\theta/2) & \sin(\theta/2)\\ -\sin(\theta/2) & \cos(\theta/2)\end{pmatrix}\in SL(2,\mathbb{R})$ and the linear map $f_A(z)$. This map satisfies the condition of Schwarts lemma again and since $f_A(i)=i$ and $f'_A(i)=e^{i\theta}$ the composite $G\circ f_A\circ G^{-1}(z)=e^{i\theta}=G\circ f\circ G^{-1}$ and thus $f_A=f$.

Finally let $z_0\in\mathbb{H}$. Consider the matrix $M=\begin{pmatrix}\sqrt{im(z_0)} & 0\\ 0 & 1/\sqrt{im(z_0)}\end{pmatrix}$ this matrix satisfies $f_M(i)=im(z_0)$ and the matrix $N=\begin{pmatrix}1 & z_0-f_M(i)\\ 0 & 1\end{pmatrix}$ induces a translation map in $\mathbb{H}$ that sends $f_M(i)$ to z_0. Then the composite $h=f_N\circ f_M=f_{NM}$ is an automorphism of $\mathbb{H}$ that sends i to z_0.

Now we will prove that for any $f\in\mathrm{Aut}(X)$ there is a matrix $A\in SL(2,\mathbb{R})$ such that $f=f_A$. For such f there is a z_0 such that $f(z_0)=i$, and from the above construction, there is an $h\in\mathrm{Aut}(X)$ such that $h(i)=z_0$. The composite $h\circ f$ is an automorphism of $\mathbb{H}$ that sends i to i and thus $h\circ f=f_B$ for some $B\in SL(2,\mathbb{R})$ using the earlier construction. Since $f=h^{-1}\circ f_B$ and h is also of the form f_C we get $f=f_{C^{-1}}\circ f_B=f_{C^{-1}B}=f_A$. □

The automorphism group of simply connected Riemann surface is infinite; in particular, since any compact Riemann surface of genus $g=0$ is isomorphic to $\mathbb{P}^1$ we conclude that $\mathrm{Aut}(X)$ is an infinite group for the case of genus zero.

Now let X be a simply connected Riemann surface and $f\in\mathrm{Aut}(X)$. Denote by $\mathrm{Fix}_X(f)$ the set of elements $z\in Z$ such that $f(z)=z$, since $f(z)=\frac{az+b}{cz+d}$, any fixed point of f satisfies the equation $z=\frac{az+b}{cz+d}$, or equivalently, $cz^2+(d-a)z-b=0$, therefore $|\mathrm{Fix}_X(f)|\leq 2$ if $f\neq id$. This is not a coincidence, and actually for any Riemann surface X and any $f\in\mathrm{Aut}(X)$ the number of fixed points of f is finite and it is bounded by the genus of X, as we will illustrate in the following result.

Theorem 2.3. *Let X be a compact Riemann surface of genus g and $f\in\mathrm{Aut}(X)$. If $f\neq id$ then $|\mathrm{Fix}_X(f)|\leq 2g+2$.*

Proof. Let $p\in X$ such that $f(p)\neq p$. From Proposition 1.3, there exist a meromorphic function $g:X\to\mathbb{P}^1$ that has a pole of order $r\leq g+1$ at p and it is holomorphic in $X\smallsetminus\{p\}$. Consider the meromorphic map $h(z)=g(z)-g(f(z))$ and notice that for any $x\in\mathrm{Fix}_X(f)$, $h(x)=0$ and since the zeroes of h are finite by compactness, we conclude that $\mathrm{Fix}_X(f)$ is finite. Moreover, as g has a single pole of order r at p, $g\circ f$ has a single pole of order r at $f^{-1}(p)$; thus h has two poles of combined order at most $2r\leq 2g+2$. Since any meromorphic function has the same number of zeros and poles, and $\mathrm{Fix}_X(f)$ is a subset of the zeros of h, we conclude that $|\mathrm{Fix}_X(f)|\leq 2g+2$. □

3 Automorphism of complex tori

Let X be a compact Riemann surface of genus $g=1$, recall that a consequence of the Abel's theorem [Forster, 1991, §20,21] is that X is isomorphic to a complex torus $\mathbb{C}/\Lambda$. Observe that any complex torus inherits a group structure under the addition, so the translation by elements $x\in X$ is an automorphism of X. This shows that for surfaces of genus $g=1$, $\mathrm{Aut}(X)$ is infinite. However, in this section we will describe explicitly the automorphism group of a complex torus $X = \mathbb{C}/\Lambda$ for Λ a lattice in $\mathbb{C}$.

Theorem 3.1. *Let X be a compact Riemann surface of genus $g = 1$, from the uniformization theorem, $X \cong \mathbb{C}/\Lambda$. Then $\mathrm{Aut}(X)$ is infinite and $\mathrm{Aut}(X) \cong X \times G$ where G is a cyclic group of order either $2, 4$ or 6.*

Proof. We will proof this statement in several steps.

1. Any $f \in \mathrm{Aut}(X)$ is of the form $f(z \mod \Lambda) = az + b \mod \Lambda$ and $a\Lambda = \Lambda$ and a is a n-th root of unity of some n.

   Let $\pi : \mathbb{C} \to X$ be the projection map which is a covering map. Since f is unramified (from the Riemann-Hurwitz theorem), f is a covering and so the composite $f \circ \pi$ is a covering map. Recall that $\mathbb{C}$ is the universal covering of X and thus there is a map $F : \mathbb{C} \to \mathbb{C}$ that makes the diagram

   $$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & X \end{array}$$

   commutative. Moreover, from uniqueness of the universal covering, $F \in \mathrm{Aut}(\mathbb{C})$ and from Proposition 2.1, $F(z) = az + b$ for some $a, b \in \mathbb{C}$ and $a \neq 0$. The commutative of the above diagram imply that $a\Lambda \subseteq \Lambda$ and $f(z \mod \Lambda) = az + b \mod \Lambda$. The reverse inclusion follows from considering f^{-1}. Now it only remains to show that $|a| = 1$. In fact, let $\lambda \in \Lambda \setminus \{0\}$ of minimal length, now since $a\Lambda = \Lambda$ we get that $|\lambda| \leq |a\lambda|$ and thus $|a| \leq 1$. The reverse inequality holds by considering a^{-1}. Now notice that $S = \{\gamma \in \Lambda : |\gamma| = |\lambda|\}$ is finite, from the above remark we see that F permutes all the elements of S and thus $F^n|_S$ is the identity map for some $n \geq 1$. This implies that $a^n = 1$ and so a is n-root of unity.

2. $\mathrm{Aut}(X) \cong \mathrm{Aut}_t(X) \rtimes \mathrm{Aut}_0(X)$ where $\mathrm{Aut}_t(X)$ are the translation automorphisms, i.e., the maps of the form $f(z) = z + b$ for $b \in X$ and $\mathrm{Aut}_0(X)$ are the automorphism of X fixing 0. It is easy to see that both $\mathrm{Aut}_t(X)$ and $\mathrm{Aut}_0(X)$ are subgroups of $\mathrm{Aut}(X)$.

   From the previous item, any $f \in \mathrm{Aut}(X)$ is of the form $f(z) = az + b$ for some a, b with $|a| = 1$ and $b \in X$. Then the map $\Phi : \mathrm{Aut}(X) \to \mathrm{Aut}_0(X)$ given by $\Phi(f)(z) = f(z) - f(0) = az$ is a surjective group homomorphism whose kernel is precisely $\mathrm{Aut}_t(X)$ and so it is a normal subgroup of $\mathrm{Aut}(X)$. Also, since in general $f \circ g \neq g \circ f$ for $f \in \mathrm{Aut}_t(X)$ and $g \in \mathrm{Aut}_0(x)$ we conclude that $\mathrm{Aut}(X) \cong \mathrm{Aut}_t(X) \rtimes \mathrm{Aut}_0(X)$.

   Finally, notice that the map $b \mapsto f(z) = z+b$ is an isomorphism of groups from X and $\mathrm{Aut}_t(X)$, this implies that $\mathrm{Aut}(X) \cong X \rtimes \mathrm{Aut}_0(X)$.

3. $\mathrm{Aut}(X)$ is an infinite group; however, $\mathrm{Aut}_0(X)$ is finite and it is either isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6Z$.

   The first claim follows easily since there are infinitely many translation automorphism on X. Now let $f \in \mathrm{Aut}_0(X)$, that is $f(z) = az$ with $a\Lambda = \Lambda$ and $|a| = 1$. If $a = \pm$, $f(z)$ is either the identity map or the group inversion of X respectively, these maps always exist in $\mathrm{Aut}_0(X)$ and so $|\mathrm{Aut}_0(X)| \geq 2$. Assume now that a is not real and let $\lambda \in \Lambda \setminus \{0\}$, then λ and $a\lambda$ generate the lattice Λ and since $a^2\lambda \in \Lambda$, we get an equation $a^2\lambda = ma\lambda + b\lambda$. Dividing out by λ we obtain a quadratic equation with integers coefficients $a^2 - ma - b = 0$. As $|a| = 1$ we conclude that a is a root of unit that satisfies a quadratic equation, that is, a is either a $3rd$, $4th$ or $6th$ root of unit. In the first and later situation, we have $\text{Aut}_0(X) \cong \mathbb{Z}/6\mathbb{Z}$, and in the second situation $\text{Aut}_0(X) \cong \mathbb{Z}/4\mathbb{Z}$. If there is no such complex number a, we obtain $\text{Aut}_0(X) \cong \mathbb{Z}/2$.

The last isomorphism has a geometric meaning: Recall that λ and $a\lambda$ are generators of Λ of the same length if a is non-real. If $\text{Aut}_0(X) \cong \mathbb{Z}/4\mathbb{Z}$ then $a = \pm i$ and thus λ and $i\lambda$ form an angle of $\pi/2$ in the plane, that is, Λ is an square lattice. In the case of $\text{Aut}_0(X) \cong \mathbb{Z}/6\mathbb{Z}$, the generators form an angle of $\pi/3$ and thus Λ is an hexagonal lattice. □

4 Hurwitz automorphism theorem

In 1878 Schwartz proved that for any Riemann surface X of genus $g \geq 2$, the group $\text{Aut}(X)$ is finite. Later, in 1894 Hurwitz [Hurwitz, 1892] found an upper bound for the order of $\text{Aut}(X)$ which depends on the genus of X. We will recreate his proof that follows from the Riemann-Hurwitz theorem and a combinatorial argument on the ramification index of the possible ramification points that can show up in the formula.

Let X be a CRS and G be a finite group acting by automorphism of X, that is, the map $l_g(x) = g \cdot x$ is a biholomorphism. In the case of smooth manifolds or without assuming compactness, we need extra conditions (properly discontinuous action for example) on the action to make the orbit space X/G a smooth manifold; however, the rigidity of compact Riemann surfaces assures that we do not need to assume extra conditions and actually X/G is a Riemannn surface and the projection map $\pi : X \to X/G$ is a branched covering.

Proposition 4.1. *Consider a group action of a finite group G on a CRS X, for any $x \in X$, denote by $G_x = \{g \in G : g \cdot x\}$ the stablizer subgroup and $S = \{x \in X : |G_x| > 1\}$ is finite*

Proof. Since X is compact, we only need to show that S is discrete. We will show that X has no accumulation points. In fact, let $x \in X$ an accumulation point and $x_n \to x$ a sequence of elements $x_n \in S$ converging to x. For each x_n there is a non trivial element $g_n \in G_{x_n}$. As G is finite, there is a $g \in G$ such that $g = g_n$ for infinitely many values of n; so we can assume that $g = g_n$ for every n. Therefore, we obtain $g \cdot x = g \lim_n x_n = \lim_n g \cdot x_n = \lim_n x_n = x$. Therefore, we get an infinite set with an accumulation point where the maps l_g and $l_e = id$ coincide. From the identitiy theorem in complex analysis we obtain that $g = e$ which is a contradiction. □

Theorem 4.2. *The complex structure of X induces a complex structure on the orbit space X/G such that the projection map $\pi : X \to X/G$ is a holomorphic map of degree $|G|$ and for any $x \in X$, the ramification index $\nu_x(\pi) = |G_x|$.*

Proof. It will follows from the existence of a neighborhood $U \subseteq X$ around any $x \in X$ satisfying the following properties.

- $gU = U$ for any $g \in G_x$.
- $U \cap gU = \varnothing$ for any $g \notin G_x$.
- The induced map $\alpha : U/G_x \to X/G$ is a homeomorphism onto an open subset of X/G.
- If $y \in U$ and $g \cdot y = y$ for some $g \in G_x$, then $y = x$.

□

Let $y \in X/G$ be a ramification point, then $\pi^{-1} = \{x_1, \dots, x_s\}$ are in the same G-orbit and so their stabilizers are conjugate subgroups, thus they have the same order r. From the orbit stabilzer theorem we get that $s = |G|/r$. Now we can derive Hurwitz bound on the order of $|G|$ and it is a direct consequence of the Riemann-Hurwitz theorem.

Theorem 4.3 (Hurwitz,1893)**.** *Let G be a finite group acting by automorphism on a compact Riemann surface X of genus $g \geq 2$. Then $|G| \leq 84(g-1)$.*

Proof. Let $Y = X/G$ and g' denote the genus of Y. Y has only finitely many ramification points $y_1, \dots, y_n$ and $\pi^{-1}(y_i) = \{x_1^i, \dots, x_{s_i}^i\}$. From the Riemann-Hurwitz theorem we get

$$\begin{aligned} 2g-2 &= |G|(2g'-2) + \sum_{i=1}^{n}\sum_{j=1}^{s_i} \nu_{x_j^i}(\pi) - 1 \\ &= |G|(2g'-2) + \sum_{i=1}^{n}\sum_{j=1}^{s_i} r_i - 1 \\ &= |G|(2g'-2) + \sum_{i=1}^{n} s_i(r_i - 1) \\ &= |G|(2g'-2) + \sum_{i=1}^{n} \frac{|G|}{r_i}(r_i - 1) \\ &= |G|\left(2g'-2+\sum_{i=1}^{n} 1 - \frac{1}{r_i}\right) = |G|(2g'-2+R). \end{aligned}$$

Note that if $R \neq 0$, $R \geq 1/2$ since $r_i \geq 2$. Now we analyze the possible values of g' and R to derive the desired bound.

- Case $g' \geq 1$. If $R = 0$, the formula turns into $2(g-1) = |G|(2g'-2)$. Since $g > 1$ and $|G| > 1$ we must have $g' \geq 2$ and thus $(2g'-2) \geq 2$. So we obtain $|G| \leq g-1$. Now if $R \neq 0$, we have that $R \geq 1/2$ and thus $2(g-1) \geq |G|/2$ and so $|G| \leq 4(g-1)$.
- Case $g' = 0$. The equation turns into $2g-2 = |G|(-2+R)$; and since the left hand side is positive, we must have $R > 2$. Now we need to look at the number of ramification points. If $n \geq 5$, then $R \geq n/2 \geq 5/2$ and so $|G| \leq 4(g-1)$. Now assume $n = 4$. If $r_i = 2$ for $i = 1,2,3,4$ we get $R = 2$ which is impossible. So we must have at least $r_1 \geq 3$ and $r_i \geq 2$ for $i = 2,3,4$. Then we get $R \geq 2/3 + 3(1/2) = 13/6$, so we obtain $|G| \leq 12(g-1)$. Now we look at $n = 3$, again we must have at least $r_1 \geq 3$ and $r_2, r_3 \geq 2$. If $(r_1, r_2, r_3) = (3,3,3)$, $(3,2,6)$ or $(3,4,4)$ then $R = 2$ which is impossible and these are the only cases where $R = 2$. So we need to assume that $R > 2$. Since $r_1 \geq 3$ we get that $(1 - 1/r_2) + (1 - 1/r_3) > 2 - 2/3 = 4/3$. If we let $r_2 \geq 2$ we get $(1 - 1/r_3) > 4/3 - 1/2 = 5/6$, in this case we obtain $r_3 \geq 7$; finally we obtain $R \geq 2/3 + 1/2 + 6/7 = 2\frac{1}{42}$. Moreover, this is the minimum value that R can attach and so $|G| \leq 84(g-1)$.

Hurwitz knew that its bound is sharp. In fact, in 1879 Klein [Klein, 1878] studied a Riemann surface of genus 3 with exactly 184 automorphism. So he developed some criterion to determine when the action of a finite group G, $|G|$ attains the bound. However, he left open the question whether there exists other Riemann surfaces that does attain the bound. ☐

Definition 4.4. We say that a finite group G is a Hurwitz group if it acts by automorphism on a CRS X of genus g and $|G| = 84(g-1)$.

Hurwtiz proved the following condition for a group G to be a Hurwitz group.

Proposition 4.5. *G is a Hurwitz group if and only if it is generated by some elements $x, y \in X$ that satisfy the relations $x^2 = y^3 = (xy)^7 = e$ (and possibly other relations).*

A lower bound for $|\mathrm{Aut}(X)|$ was discovered independently by Accola [Accola, 1968] and Maclachlan [Maclachlan, 1969]. They constructed a group of order $8g + 8$ acting on a Riemann surface of genus g and showed that this is the minimum possible order of a finite group action on X. So we can summarize

Theorem 4.6. *Let X be a compact Riemann surface of genus $g \geq 2$. Then $8g + 8 \leq |\mathrm{Aut}(X)| \leq 84(g-1)$.*

Klein showed by explicit computation that if $g = 2$, $|\mathrm{Aut}(X)| \leq 48$. Later, Gordan showed that if $g = 4$, $|\mathrm{Aut}(X)| \leq 120$ and Wiman showed that for $g = 5$ and $g = 6$ we have $|\mathrm{Aut}(X)| \leq 192$ and $|\mathrm{Aut}(X)| \leq 420$ respectively. Therefore, for low-genus Riemann surfaces, the bound is not attained except when $g = 3$ since the Klein-quartic does attain the bound. Moreover, Macbeath [Macbeath, 1961] showed that there are infinitely many values of g such that a Hurwitz group of order $84(g-1)$ is realized. On the other hand, in the same paper, Accola and Maclachaln constructed infinitely many values of g with a compact Riemann surface of genus g having exaclty $8g + 8$ automorphisms.

5 $\mathrm{Aut}(X)$ is finite

In this section we will provide a skectch of several proofs of the fact that $\mathrm{Aut}(X)$ is a finite group for X a compact Riemann surface of genus $g \geq 2$. First, we will illustrate Schwartz proof using Weierstrass points; then we will show how the Fuschian groups and Hyperbolic geometry implies Hurwitz automorphism theorem and finally we will show that there is a faithful representation of $\mathrm{Aut}(X)$ onto the isomorphism of the space of holomorphic 1-forms of X $\Omega^1(X)$. For further details on the proofs the Farkas's book [Farkas and Kra, 1992] is the followed reference.

From the rest of this section we will assume that X is a compact Riemann surface of genus $g \geq 2$.

Weierstrass points

Let us start with the following motivation. Let X be a CRS and $f \in \mathcal{M}(X)$ a meromorphic function with $\mathrm{Ord}_p(f) = -1$ and holomorphic else where. Then f induces a biholomorphism $X \cong \mathbb{P}^1$. The existence of such f is equivalent to $l(D) > 1$ where $D = 1 \cdot p$ is a divisor. Therefore, if X has genus greater than zero, such function f can not exist.

Now assume X is a CRS of genus $g > 0$, we might ask now, for which values of n does not exist $f \in \mathcal{M}(X)$ with $\mathrm{Ord}_p(f) = -n$ and holomorphic elsewhere? this is equivalent to ask $l(D_n) < 2$ where $D_n = n \cdot p$ is a divisor.

From the Riemann Roch theorem 1.2 we get

$$l(D_n) - l(K - D_n) = \deg(D_n) + 1 - g$$

If we want $l(K-D)$, we may require $\deg(K-D_n) < 0$, or equivalently, $n > 2g-2$ since $\deg(K) = 2g-2$. Therefore, for $n \geq 2g-1$, $l(D_n) = n + 1 - g$; moreover, if $n \geq 2g$, $l(D_n) \geq g + 1 \geq 2$ and so the required function f exists.

Notice that $l(D_{j+1}) \le l(D_j) + 1$ and then in the sequence

$$1 = l(D_0) \le l(D_1) \le \cdots l(D_{2g}) = g + 1$$

the increasing $2g$-steps are of size at most 1 and so there exists integers $1 = n_1, i \ldots, n_g$ such that $l(D_{n_i}) = l(D_{n_i - 1})$. In this case, there is no function $f \in \mathcal{M}(X)$ with $\mathrm{Ord}_p(f) = n_i$. These numbers are called *gap numbers of* p. If n is a gap number of p, then $l(D_{n-1}) - l(D_n) = 0$. Using the Riemann-Roch formula in each term, the equation turns into $l(K - D_{n-1}) - l(K - D_n) = 1$, or equivalently, there exists a holomorphic differential ω on X such that $\mathrm{Ord}_p(\omega) = n - 1$.

Now let $V = \Omega^1(X)$ be the space of holomorphic differentials. Recall that $\dim V = g$, and by induction, there is a basis of V that locally is locally given by holomorphic functions φ_j, $j = 1, \ldots, g$ satisfying

$$\mathrm{Ord}_p(\varphi_1) < \mathrm{Ord}_p(\varphi_2) < \cdots < \mathrm{Ord}_p(\varphi_g)$$

(Let $d_1 = \min\{\mathrm{Ord}_p(\omega) : \omega \in V\}$ and $\omega_1 = \varphi_1 dz$ around p, write $V = \omega_1 \oplus V'$ and use induction on V'). And now define the Wronskian as

$$W = \det \begin{pmatrix} \varphi_1 & \cdots & \varphi_g \\ \varphi_1' & \cdots & \varphi_g' \\ \vdots & & \vdots \\ \varphi_1^{(g-1)} & \cdots & \varphi_g^{(g-1)} \end{pmatrix}$$

So we will prove now that

Proposition 5.1. $\mathrm{Ord}_p W = \sum_{i=1}^{g} d_j - j + 1$ *where* $d_j = \mathrm{Ord}_p(\varphi_j)$.

Proof. Write $W = \det(\varphi_1, \ldots, \varphi_g)$. Notice that $\det(f\varphi_1, \ldots, f\varphi_g) = f^g(\varphi_1, \ldots, \varphi_g)$. We will proceed now by induction on g. If $g = 1$, $W = \varphi_1$ and thus $\mathrm{Ord}_p(W) = \mathrm{Ord}_p(\varphi_1)$. Now supose that it holds for k and we will prove it for $k + 1$. In fact,

$$\det(\varphi_1, \varphi_2, \ldots, \varphi_{k+1}) = \varphi_1^{k+1} \det(1, \varphi_2/\varphi_1, \ldots, \varphi_{k+1}/\varphi_1) = \varphi_1^{k+1} \det((\varphi_2/\varphi_1)', \ldots, (\varphi_{k+1}/\varphi_1)')$$

And thus by induction we get

$$\mathrm{Ord}_p W = (k+1)d_1 + \sum_{j=2}^{k+1} (d_j - d_1 - 1) - (j - 2) = (k+1)d_1 + \sum_{j=2}^{k+1} d_j - d_1 - j + 1 = \sum_{j=1}^{k+1} d_j - j + 1$$

□

Notice that in fact, $d_j + 1 = n_j$ the j-th gap number at p. So $\mathrm{Ord}_p(W) = \sum_{j=1}^{g} n_j - j$. Combining this result with the gaps points, we have then

Theorem 5.2. *The following are equivalent.*

1. $\mathrm{Ord}_p W > 0$.
2. $W(p) = 0$.
3. *There exists a non-zero holomorphic differential* $\omega \in \Omega^1(X)$ *such that* $\mathrm{Ord}_p(\omega) \ge g$.
4. $l(K - D_g) > 0$.

5. $l(D_g) \geq 2$.

6. *The gap numbers of* p *are* $\{1, \ldots, g\}$.

Definition 5.3. A point $p \in X$ satisfying one (and all) of the above conditions is called a Weierstrass point of X.

Since the Weierstrass points are isolated, they are finite. Call the set of Weierstrass points of X $W(X)$. Now we will find a lower and upper bound for $|W(X)|$. First we need the following result.

Proposition 5.4. *W is a holomorphic m-differential where $m = \frac{g(g+1)}{2}$ and* $\deg(W) = g^3 - g$.

Proof. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $\Omega^1(X)$. Choose a chart U around p such that $\omega_j = \varphi_j(z)dz$ and let $w = f(z)$ a holomorphic change of coordinates given by the choosing of another chart V. In this chart we have $\omega_j = \psi(w)dw$ and thus in the common domain we have the relation $\psi(w) = \psi(f(z))f'(z) = \varphi_j(z)$. Therefore, we obtain

$$\det(\varphi_1, \ldots, \varphi_g) = \det(\psi_1(f)f', \psi_2(f)f', \ldots, \psi_g(f)f'$$

by elemental operations in the last matrix, we obtain

$$W = \det\begin{pmatrix} \psi_1(f)f' & \cdots & \psi_g(f)f' \\ \psi_1'(f)(f')^2 & \cdots & \psi_g'(f)(f')^2 \\ \vdots & & \vdots \\ \psi_1^{(g-1)}(f)(f')^g & \cdots & \psi_g^{(g-1)}(f)(f')^g \end{pmatrix}$$

And finally we obtain $\det(\varphi_1, \ldots, \varphi_g) = (f')^{\frac{g(g+1)}{2}} \det[\psi_1, \ldots, \psi_g](f)$ and so W is a $m = \frac{g(g+1)}{2}$ differential, and its degree is $(2g-2)m = g^3 - g$. $\square$

So we have $g^3 - g = \deg(W) = \sum_{p \in X} \operatorname{Ord}_p(W)$ and thus we obtain:

Corollary 5.5. *The number of Weierstrass points* $|W(X)| \leq g^3 - g$.

For our purposes, we need to find a lower bound. To do that, we need to look at the sequence of *non-gap numbers of* p. That is, let $m_1, \ldots, m_g$ denote the numbers that are not gap numbers in the sequence $\{1, \ldots, 2g\}$. Notice that $m_g = 2g$ and for each m_i there exist a meromorphic function f_i such that $\operatorname{Ord}_p(f_i) = m_i$. It is straightforward to check the following properties of this sequence

- For each $0 < i < g$, $m_i + m_{g-i} \geq 2g$.
- If $m_1 = 2$, then $m_i = 2i$ and $m_i + m_{g-i} = 2g$ for all i.
- if $m_1 > 2$ there is a j, $1 < j < g$ such that $m_j + m_{g-j} > 2$.

From these properties we get

Proposition 5.6. $\sum_{j=1}^{g-1} m_j \geq g(g-1)$ *with equality if and only if* $m_1 = 2$.

As a corollary we obtain

Corollary 5.7. $|W(X)| \geq 2g + 2$.

Proof. Let $p \in X$, $n_1 < \ldots < n_g$ be the gap numbers at p and $m_1 < \cdots < m_g$ the non-gap numbers at p. Then from Proposition 5.1 we have

$$\begin{aligned}
\mathrm{Ord}_p(W) = \sum_{j=1}^{g} n_j - j &= \sum_{j=1}^{2g} j - \sum_{j=1}^{g} m_j - \sum_{j=1}^{g} j \\
&= \sum_{j=g+1}^{2g-1} j - \sum_{j=1}^{g-1} m_j \\
&\leq \frac{3}{2} g(g-1) - g(g-1) = \frac{g(g-1)}{2}
\end{aligned}$$

Since $\deg(W) = g^3 - g$ there are at least $\deg(W) / \mathrm{Ord}_p(W) = 2g+2$ Weierstrass points. □

Let $F : X \to Y$ a holomorphic map between CRS. Define $F^* : Div(Y) \to Div(X)$ by setting $F^*D(p) = \nu_p(F) D(F(p))$ where $\nu_p(F)$ is the ramification index of F at p. Let $f \in L(D)$, notice that $\mathrm{Ord}_p F^*(f) = \mathrm{Ord}_p(f(F)) = \nu_p(F)\,\mathrm{Ord}_p(f) \geq -\nu_p(F) D(p) = -F^*D(p)$, that is, $f \in L(F^*D)$. So if F is an isomorphism, we have $L(D) \cong L(F^*D)$. Now we can prove

Proposition 5.8. *Let $T \in \mathrm{Aut}(X)$be a an automorphism of a CRS of genus $g \geq 2$. If $p \in X$ is a Weierstrass point, so is $T(p)$.*

Proof. If p is a Weierstrass point, the divisor $D = g \cdot p$ satisfies $l(D) \geq 2$. Notice that $T^*D = g \cdot T(p)$ and by the above Remark we obtain that $l(T^*D) = l(D) \geq 2$ and thus $T(p)$ is also a Weierstrass point. □

Now we will see that the Weierstrass points have a particular characterization when X is a hyperelliptic Riemann-Surface.

Definition 5.9. We say that X is hyperelliptic if there is a two-sheeted branched covering $f : X \to \mathbb{P}^1$.

The existence of the covering f is equivalent to the existence of a divisor D with $l(D) \leq 2$ and $\deg(D) = 2$. In particular, from the Riemann-Hurwitz theorem, the map f must have $2g+2$ branch points in $\mathbb{P}^1$.

The following result relates the Weierstrass points on a hypereliptic Riemann surface and the automorphism of X.

Proposition 5.10. *Let X be a compact Riemann surface. then*

1. *X has at least $2g+2$ Weierstrass points and it has exaclty $2g+2$ if and only if X is hyperelliptic.*
2. *X is hyperelliptic if and only if there is an automorphism $J \in \mathrm{Aut}(X)$ such that $J^2 = id$ and the fixed points of J are exactly the Weierstrass points.*
3. *If X is hyperelliptic, for any $T \in \mathrm{Aut}(X)$, $T \neq I, J$, T has at most 4 fixed points.*

Now we can sketch Schwartz arguing.

Theorem 5.11. *$|\mathrm{Aut}(X)|$ is finite for $g \geq 2$.*

Proof. Write $W(X)$ the (finite) set of Weierstrass points of X.

- If $T \in \mathrm{Aut}(X)$, $T(W(X)) = W(X)$. That is, T induces a permutation of the Weierstrass points of X by Proposition 5.8.
- If X is not hyperelliptic, from Theorem 2.3, the only automorphism that fixes all the Weierstrass points is the identity map.
- If X is hyperelliptic, any automorphism different from J and id fixes at most $4 < 2g+2$ Weierstrass points.
- Let $S(W(X))$ denote the group of permuatations of the set $W(X)$. The induced map $\Phi : \mathrm{Aut}(X) \to S(W(X))$ is injective unless X is hyperelliptic. In the latter case $\ker(\Phi) = \{id, J\}$.
- Φ is a group homomorphism of $\mathrm{Aut}(X)$ onto a finite group and it has finite kernel. Thus $\mathrm{Aut}(X)$ is finite.

□

Hyperbolic geometry

Recall that from the uniformization theorem, for a compact Riemann surface X of genus $g \geq 2$, the upper half plane $\mathbb{H}$ is its universal covering and we can realize X as the quotient $\mathbb{H}/\Lambda$ where Λ is a subgroup of $\mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R})$ that acts properly discontinuous on $\mathbb{H}$. Poincare [Poincaré, 1882] in 1882 showed that a subgroup $G \subseteq \mathrm{Aut}(\mathbb{H})$ acts properly discontinuous on $\mathbb{H}$ if and only if G is a discrete subgroup; and he introduced the notion of Fuschian group which is a discrete subgroup of $PSL(2,\mathbb{R})$. Poincare introduced this notion while he was studying the Hyperbolic plane modeled as the upper half plane $\mathbb{H}$. Even though he did not make a direct connection with the automorphism group of compact Riemann surfaces, in 1945 Siegel [Siegel, 1945] proved the finiteness of $\mathrm{Aut}(X)$ and Hurwitz bound by using the geometry of the Hyperbolic space and relating Fuschian groups with $\mathrm{Aut}(X)$.

Theorem 5.12. $\mathrm{Aut}(X)$ *is finite and* $|\mathrm{Aut}(X)| \leq 84(g-1)|$.

Proof.
1. (Siegel) For any Fuschian subgroup $G \subseteq PSL(2,\mathbb{R})$, $Area(\mathbb{H}/G) \geq \pi/21$.
2. Realize $X = \mathbb{H}/\Lambda$ where Λ is a discrete subgroup of $PSL(2,\mathbb{R})$.
3. The normalizer subgroup $\Gamma = N(\Lambda)$ of Λ in $PSL(2,\mathbb{R})$ is discrete.
4. $\mathrm{Aut}(X) \cong \Gamma/\Lambda$.
5. Consider the Riemann surface $Y = \mathbb{H}/\Gamma$ and the canonical map $\pi : X \to Y$ is a covering map of degree $[\Gamma : \Lambda]$.
6. $\frac{Area(X)}{Area(Y)} = [\Gamma : \Lambda]$.
7. From Gauss-Bonnet theorem $Area(X) = -2\pi\chi(X) = 2\pi(2g-2)$.
8. Finally we get
$$|Aut(X)| = [\Gamma : \Lambda] = \frac{Area(X)}{Area(Y)} \leq \frac{2\pi(2g-2)}{\pi/21} = 84(g-1)$$

□

Representation of $\mathrm{Aut}(X)$ on the space of holomorphic 1-forms

Recall that Ω^1 is a finite dimensional vector space of dimension g. Let $\omega \in \Omega^1$ and $T \in \mathrm{Aut}(X)$. Let $x \in X$ and choose chart around x such that locally $\omega = f(z)dz$ where f is a holomorphic map. Also choose a chart around $T(x)$ such that T^{-1} have the form $h(z)$. Consider the action of $\mathrm{Aut}(X)$ on Ω^1 given by $T \cdot \omega = (T^{-1})^* \omega$. Locally $T \cdot \omega = f(h(z))h'(z)dz$. We have then

Proposition 5.13. *1. For $T_1, T_2 \in \mathrm{Aut}(X)$ and $\omega \in \Omega^1$, $(T_1) \cdot (T_2 \cdot \omega) = (T_1 \circ T_2) \cdot \omega$.*

2. The induced map $T : \Omega^1 \to \Omega^1$ is a $\mathbb{C}$-linear isomorphism.

Theorem 5.14. *The representation of $\mathrm{Aut}(X)$ on $GL(\Omega^1)$ is faithful.*

Proof. Let $T \in \mathrm{Aut}(X)$ not the identity. Assume that X is not hyperelliptic, then there is a Weierstrass point $x \in X$ such that $T(x) \neq x$. Then consider a non-zero $\omega \in \Omega^1$ such that $ord_x(\omega) \geq g$. If $T \cdot \omega = \omega$, then ω has at least g zeros at x and g zeros at $T(x)$; thus ω has at least $2g > (2g-2)$ zeros. That is, $\omega = 0$, which is a contradiction. So T can not induce the identity map on Ω^1.

Now assume that X is hyperelliptic. It can be shown that X can be realized locally as

$$w^2 = (z - x_1) \cdots (z - x_{2g-2})$$

where $x_1, \ldots, x_{2g-2}$ are distinct complex numbers. A basis for Ω^1 is given by the forms $\omega_j = \frac{z^{j-1}dz}{w}$ for $j = 1, \ldots, g$. Now let $T \in \mathrm{Aut}(X)$, if $T \neq id, J$ the proof for non-hyperelliptic Riemann surfaces is still valid. Now assume $T = J$, then J is represented by the involution $(z, w) \mapsto (z, -w)$ and clearly it does not acts as the identity on the forms ω_j. □

The trace of the matrix representing an automorphism T in $GL(V)$ can be explictly described, the result is known as the *Eichler trace formula* and it depends on the fixed points of T. (For full reference on the details of the statement and its proof see [Farkas and Kra, 1992, Ch V.2])

A nice consequence of the Eichler trace formula is the Lefschetz fixed point formula.

Proposition 5.15. *Let t be the number of fixed points of T. Then*

$$\mathrm{Tr}(T) + \overline{\mathrm{Tr}(T)} = t$$

Notice that the matrix $T + \bar{T}$ is the representation of the automorphism into the space of Harmonic differentials of X.

6 Final Remarks

In this final section we include the Klein quartic that was the first example of a Riemann surface that attains the Hurwitz bound. In fact, this surface has been widely studied and has helped in the development of further studies in several branches of mathematics. This curve is fascinating since it can be constructed as a tiled platonic hyperbolic polygon and its automorphism group can be computed just by visual arguments as Klein showed in his first construction. The Klein quartic has books, websites, videos and even a real sculpture just for itself. A nice reference for this section and an overview of this Riemann surface see [Levy, 2001]

Consider the Projective curve X in $\mathbb{P}^2$ given by the equation

$$X^3Y + Y^3Z + Z^3X = 0.$$

From the degree-genus formula we get $g = (d-2)(d-1)/2 = 3$. And so from the Hurwitz automorphism theorem we get $|\operatorname{Aut}(X)| \leq 84(g-1) = 168$. Now we will describe explicitly the automorphism of X.

Let ξ be a primitive $7th$-root of unit and $f : \mathbb{P}^2 \to \mathbb{P}^2$ be the map given by $f([x : y : z]) = [\xi x, \xi^2 y, \xi^4 z]$. Clearly $f(X) = X$ and f defines an automorphism of order 7.

Now the cyclic permutation $g([x : y : z]) = [z : x : y])$ defines an automorphism of order 3 of X.

Finally, consider the map h induced by the matrix

$$\frac{i}{\sqrt{7}}\begin{pmatrix} \xi - \xi^6 & \xi^2 - \xi^5 & \xi^4 - \xi^3 \\ \xi^2 - \xi^5 & \xi^4 - \xi^3 & \xi - \xi^6 \\ \xi^4 - \xi^2 & \xi - \xi^6 & \xi^2 - \xi^5 \end{pmatrix}$$

it is straightforward to check that that h defines an automorphism of order 2. Now notice that $g \circ f \circ g^{-1} = f^4$ and $h \circ g \circ h^{-1} = g^2$ and consider the subgroup G of $\operatorname{Aut}(X)$ generated by f, g, h. We can check that the elements 49 elements of the form $f^m \circ h \circ f^n$ are all distinct and thus $< f >$ can not be normal in G. Notice that $2 * 3 * 7 = 42$ divides $|B|$, we get that $|G| = 42, 84, 126$ or 168. From the Sylow's theorem, we can see that in the first three cases $< f >$ is a normal subgroup of G; therefore $|G| = 168$ and thus G attains the maximum possible order for a genus 3 surface.

Also, this curve can be realized as the quotient $\mathbb{H}/\Gamma$ where Γ is a discrete subgroup of $\operatorname{Aut}(\mathbb{H})$. Similar to the case of tori, which is constructed as identifying the fundamental region given by the lattice (and usually it is a polygon), a fundamental region for the Klein quartic has the form

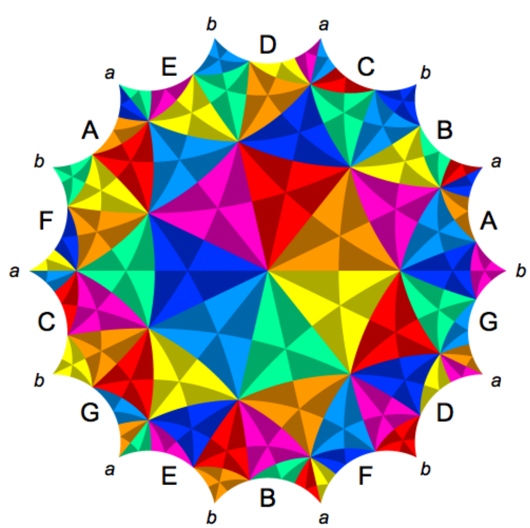

Figure 1: Fundamental region for the Klein Quartic

Gluing together the same labeled edges and vertices, we obtain a representation of the genus 3 surface in $\mathbb{R}^3$.

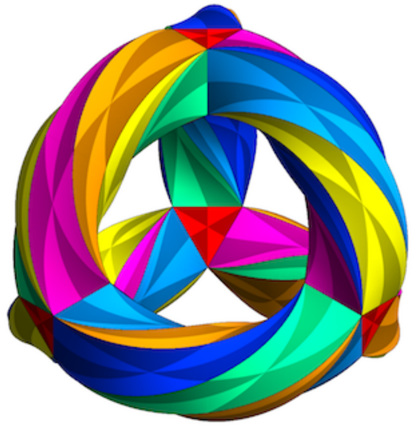

Figure 2: The Klein quartic in the space

Remark:
The figures are Copyright ©Greg Egan, *http://www.gregegan.net/SCIENCE/KleinQuarticKleinQuartic.html*.

We finish this document with the following table, which show the genus where there exist a surface (and even how many up to biholomorphism) that realizes the maximum possible order of the automorphism group.

Group	Genus	Number of surfaces
PSL(2,7)	3	1
PSL(2,8)	7	1
PSL(2,13)	14	3
PSL(2,27)	118	1
PSL(2,29)	146	3
PSL(2,41)	411	3
PSL(2,43)	474	3
J1	2091	7
PSL(2,71)	2131	3
PSL(2,83)	3404	3
PSL(2,97)	5433	3
J2	7201	5
PSL(2,113)	8589	3
PSL(2,125)	11626	1

Table 1: Simple Hurwitz groups of order less than 10^6

Here $PSL(2,m)$ denotes the group $PSL(2,\mathbb{Z}/m\mathbb{Z})$ and J_1, J_2 denote the first two *Janko Groups*.

References

[Accola, 1968] Accola, R. D. (1968). On the number of automorphisms of a closed riemann surface. *Transactions of the American Mathematical Society*, pages 398–408.

[Farkas and Kra, 1992] Farkas, H. M. and Kra, I. (1992). Riemann surfaces. In *Riemann surfaces*, pages 9–31. Springer.

[Forster, 1991] Forster, O. (1991). Lectures on riemann surfaces, volume 81 of graduate texts in mathematics.

[Hurwitz, 1892] Hurwitz, A. (1892). Über algebraische gebilde mit eindeutigen transformationen in sich. *Mathematische Annalen*, 41(3):403–442.

[Klein, 1878] Klein, F. (1878). Ueber die transformation siebenter ordnung der elliptischen functionen. *Mathematische Annalen*, 14(3):428–471.

[Levy, 2001] Levy, S. (2001). *The eightfold way: the beauty of Klein's quartic curve*, volume 35. Cambridge University Press.

[Macbeath, 1961] Macbeath, A. M. (1961). On a theorem of hurwitz. *Glasgow Mathematical Journal*, 5(2):90–96.

[Maclachlan, 1969] Maclachlan, C. (1969). A bound for the number of automorphisms of a compact riemann surface. *Journal of the London Mathematical Society*, 1(1):265–272.

[Poincaré, 1882] Poincaré, H. (1882). Théorie des groupes fuchsiens. *Acta mathematica*, 1(1):1–62.

[Siegel, 1945] Siegel, C. L. (1945). Some remarks on discontinuous groups. *Annals of mathematics*, pages 708–718.