Cohomology of $H^*_{S^1 \rtimes \mathbb{Z}/2}(S^2)$

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July 22, 2018

Let $G = S^1 \rtimes \mathbb{Z}/2$ and consider the action of G over $X = S^2$ induced by the actions of S^1 given by the rotation along the z-axis and the action of $\mathbb{Z}/2$ given by the reflection τ along the plane xz; observe that the fixed point subspace $X^G = \{N, S\}$ consist of the two poles, and $X^\tau \cong S^1$.

Throughout this document $H^*(.)$ will denote the singular cohomology ring $H^*(., \mathbb{Z}/2)$ and X_G the Borel construction associated to the action G. First we will compute the G-equivariant cohomology of S^2, and we will show that it is a free module over $H^*(BG)$. Moreover, if we restrict to $X^\tau \cong S^1$ under the action of $K = \{g \in S^1 : g^2 = 1\} \cong \mathbb{Z}/2$, we will show that $H^*_K(X^\tau)$ is also a free module over $H^*(BK)$.

We will be using mainly the Mayer-Vietoris sequence for equivariant cohomology that we illustrate in the following proposition

Proposition 1. *Let G be a topological group, let X be a G-space and $U, V \subseteq X$ subspaces such that $int(U) \cup int(V) = X$ and $G \cdot U \subseteq U$, $G \cdot V \subseteq V$. Then there is long exact sequence of abelian groups*

$$0 \to H^0_G(X) \to H^0_G(U) \oplus H^0_G(V) \to H^0_G(U \cap V) \to$$
$$H^1_G(X) \to H^1_G(U) \oplus H^1_G(V) \to H^1_G(U \cap V) \to H^2_G(X) \to \cdots$$

Proof. Observe that there is a homeomorphism $X_G \cong int(U_G) \cup int(V_G)$ induced by the decomposition

$$\begin{aligned} EG \times X &= EG \times (int(U) \cup int(V)) \\ &= (EG \times int(U)) \ \cup \ (EG \times int(V)) \\ &= int(EG \times U) \ \cup \ int(EG \times V) \end{aligned}$$

and then apply the regular Mayer-Vietoris sequence for singular cohomology. $\square$

Also, the following fact will be useful for our purposes

Proposition 2. *Let X be a G-space and $x \in X$ any elements. Denote by $G \cdot x = \{g \cdot x : g \in G\}$ the orbit space of x, and $G_x = \{g \in G : g \cdot x = x\}$ the isotropy group of x. Then under the restriction of the action of X to $G \cdot x$, there is an isomorphism*

$$H^*_G(G \cdot x) \cong H^*(BG_x).$$

Proof. There is a homeomorphism

$$\varphi : EG/G_x \to (G \cdot x \times EG)/G$$

given by $\varphi([t]) = [x.t]$; indeed,

1. φ is well defined: If $t = gs$ with $g \in G_x$, then $\varphi([t]) = [x, t] = [x, gs] = [g^{-1}x, s] = [x, s] = \varphi([s])$.

2. φ is continuous: φ is the induced map of the G_x-invariant composite

$$EG \to G \cdot x \times EG \to (G \cdot x \times EG)/G$$

   where the first map is the inclusion $t \mapsto (x, t)$ and the second map is the quotient map.

3. φ has an inverse: The map $\theta : (G \cdot x \times EG)/G \to EG/G_x$ given by $[gx, t] = [g^{-1}t]$ is its inverse.

Therefore, we have an induced isomorphism in cohomology

$$H^*(BG_x) \cong H^*(EG/G_x) \cong H^*((G \cdot x \times EG)/G) \cong H^*_G(G \cdot x)$$

□

Now we go back to the main point of this document, set $X = S^2$, $U = S^2 - \{S\}$ and $V = S^2 - \{N\}$ we have G-homotopies $U \simeq V \simeq \{*\}$. So we get that $H^*_G(U) \cong H^*_G(V) \cong H^*(BG)$. Also, $U \cap V$ is G-homotopic to S^1, the equator circle of S^2, which is the orbit space of the point $x = (1, 0, 0)$; in this case we have, $G \cdot x \cong S^1$ and $G_x = \{(1, 1), (e^{i\pi}, \tau)\} \cong \mathbb{Z}/2$. From proposition 2 it follows that $H^*_G(U \cap V) \cong H^*(BG_x) \cong H^*(B\mathbb{Z}/2)$.

The inclusion map $i : U \cap V \to U$ is G-homotopic with the map $S^1 \to \{*\}$. Therefore, there is a commutative diagram

$$\begin{array}{ccc} H^*_G(U) & \xrightarrow{i^*} & H^*_G(U \cap V) \\ \cong \downarrow & & \downarrow \cong \\ H^*(BG) & \xrightarrow{\iota^*} & H^*(BG_x) \end{array}$$

where the map in the bottom row is induced by the inclusion $\iota : G_x \to G$.

Remark 3. Recall that there is an isomorphism $H^*(BG) \cong H^*(B\mathbb{Z}/2) \otimes H^*(BS^1) \cong \mathbb{Z}/2[w,c]$ where $|w| = 1$, $|c| = 2$, such that the maps in cohomology induced by the inclusion $G_x \to G$ and projection $G \to \mathbb{Z}/2$ coincides with the canonical maps $\mathbb{Z}/2[w,c] \to \mathbb{Z}/2[w]$ and $\mathbb{Z}/2[w] \to \mathbb{Z}/2[w,c]$ respectively.

Under this remark, the map i^* coincides with the canonical map $\mathbb{Z}/2[w,c] \to \mathbb{Z}/2[w]$. Using the same argument, we also have that the map $j^* : H^*_G(V) \to H^*_G(U \cap V)$ coincides with the canonical map $\mathbb{Z}/2[w,c] \to \mathbb{Z}/2[w]$.

From the Mayer-Vietoris sequence for equivariant cohomology (Proposition 1) we get a long exact sequence of groups

$$0 \to H^0_G(X) \to H^0_G(U) \oplus H^0_G(V) \to H^0_G(U \cap V) \to H^1_G(X) \to H^1_G(U) \oplus H^1_G(V) \to$$
$$H^2_G(X) \to H^2_G(U) \oplus H^2_G(V) \to H^2_G(U \cap V) \to \cdots$$

Recall that the map $H^k_G(U) \oplus H^k_G(V) \to H^k_G(U \cap V)$ is given by $i^* - j^*$ (or in this case by $i^* + j^*$), and by the Remark 3 such map is clearly surjective.

Therefore; there is a short exact sequence of graded $H^*(BG)$-modules

$$0 \to H^*_G(X) \to H^*_G(U) \oplus H^*_G(V) \to H^*_G(U \cap V) \to 0$$

that is,

$$H^*_G(S^2) \cong \ker(\mathbb{Z}/2[w,c] \oplus \mathbb{Z}/2[w,c] \xrightarrow{i^*+j^*} \mathbb{Z}/2[w])$$

We assert that $H^*_G(X)$ is a free module over $H^*(BG)$; as we illustrate under the next result:

Proposition 4. $\ker(i^* + j^*)$ *is freely generated by* $\{(1,1),(c,0)\}$ *as* $H^*(BG)$*-module.*

Proof. Recall that the module structure of $H^*(BG)$ over $H^*(BG) \oplus H^*(BG)$ is given by

$$r \cdot (p,q) = (rp, rq)$$

for any $r, p, q \in \mathbb{Z}/2[w,c]$. Also, the map $i^* + j^*$ is given by

$$(i^*, j^*)(p(w,c), q(w,c)) = p(w,0) + q(w,0)).$$

It is clear that $(1,1),(c,0) \in \ker(i^* + j^*)$. Now let $(p(w,c), q(w,c)) \in \ker(i^* + j^*)$; suppose that $|p| = m$ and $|q| = n$ and assume without loss of generality that $m \leq n$. Write

$$(p(w,c), q(w,c)) = (p_0, q_0) + \cdots + (p_m, q_m) + (0, q_{m+1}) + \cdots (0, q_n)$$

where p_k, q_k are homogeneous polynomials in $\mathbb{Z}/2[w,c]$ of degree k. For $m < k \leq n$, we have that $(i^* + j^*)(0, q_k(w,c)) = q(w,0) = 0$; this implies that $n \geq 2$ and $q(w,c) = c\widetilde{q_k}(w,c)$; thus we can write

$$(0, q_k) = \widetilde{q_k} \cdot (0,c) = c\widetilde{q_k} \cdot (1,1) + \widetilde{q_k} \cdot (c,0);$$

that is, $(0, q_k)$ belongs to the module generated by $\{(1,1), (c,0)\}$.

We can assume then that p, q are homogeneous polynomials in $\mathbb{Z}/2[w, c]$ of the same degree m; namely,

$$p(w,c) = \gamma w^m + c\widetilde{p}(w,c)$$

and

$$q(w,c) = \gamma' w^m + c\widetilde{q}(w,c)$$

where $|\widetilde{p}|, |\widetilde{q}| < k$. Since $(i^* + j)^*(p,q) = 0$ we conclude that $\gamma = \gamma'$; write

$$\begin{aligned}(p,q) &= \gamma w^m \cdot (1,1) + \widetilde{p} \cdot (c,0) + \widetilde{q} \cdot (0,c) \\ &= \gamma w^m \cdot (1,1) + \widetilde{p} \cdot (c,0) + c\widetilde{q} \cdot (1,1) + \widetilde{q} \cdot (c,0) \\ &= (\gamma w^m + c\widetilde{q}) \cdot (1,1) + (\widetilde{p} + \widetilde{q}) \cdot (c,0)\end{aligned}$$

which proves the assertion. □

Now we restrict to $X^\tau \cong S^1$ under the action of $K = \{g \in S^1 : g^2 = 1\} \cong \mathbb{Z}/2$. This action coincides with the reflection of a circle along the vertical axis.

With a decomposition similar to the chosen in the above case of S^2, we apply the equivariant Mayer-Vietoris sequence with $X = S^1$, $U = S^1 - \{S\} \cong \{*\}$, $V = S^1 - \{N\} \cong \{*\}$ and $U \cap V$ is homotopic to a two-points subspace. Therefore, we get

$$H^*_K(U) \cong H^*_K(V) \cong H^*(BK) \cong \mathbb{Z}/2[w]$$

where $|w| = 1$ and

$$H^*_K(U \cap V) \cong H^*(U \cap V/K) \cong H^*(\{*\}) \cong \mathbb{Z}/2.$$

From the sequence given by Proposition 1 we get

$$0 \to H^0_K(X^\tau) \to H^0_K(U) \oplus H^0_K(V) \to H^0_G(U \cap V) \to H^1_K(X^\tau) \to \cdots$$

Since $H^i_K(U \cap V) = 0$ for $i \geq 2$, we have an isomorphism

$$H^i(X) \cong H^i(U) \oplus H^i(V) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

and for the loweer lever terms we get a short exact sequence

$$0 \to H^0_K(X) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \to H^1_K(X) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0$$

The surjectivity of the map $\mathbb{Z}/2 \oplus Z/2 \to \mathbb{Z}/2$ implies that

$$H^0_K(X) \to \mathbb{Z}/2 \text{ and } H^1_K(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

Therefore, there is a short exact sequence of graded $H^*(BK)$-modules

$$0 \to H^*_K(X^\tau) \to H^*(BK) \oplus H^*(BK) \to H^*(\{*\}) \to 0$$

Recall that the module structure over $H^*(\{*\})$ is given by $w \cdot 1 = 0$; so $H^*_K(X^\tau)$ is isomorphic to the free submodule of $H^*(BK) \oplus H^*(BK)$ generated by $(1,1)$ and $(w,0)$.