Equivariant Cohomology

A topological and an algebraic approach.

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Motivation

Let X be a topological space with a continuous action of a group G. (G-space for short)

- Is there an algebraic invariant of X that captures both the topology and the nature of the action?
- The singular cohomology $H^*(X)$ depends just on the topology of X.

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How about the cohomology of the orbit space $H^*(X/G)$?

Example

Let $X = S^2$ and $G = \mathbb{Z}/2$ be the antipodal action on X. Then $H^*(X/G) \cong H^*(\mathbb{R}P^2)$.

However...

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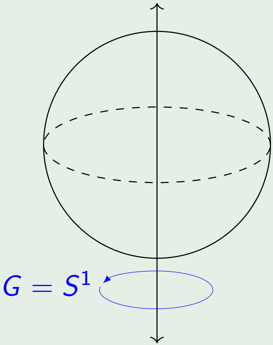

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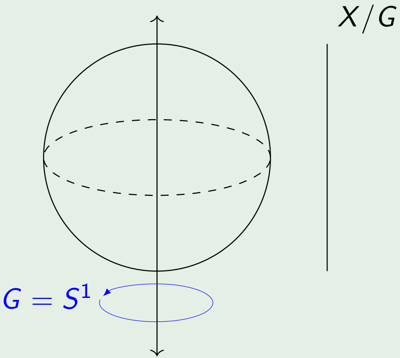

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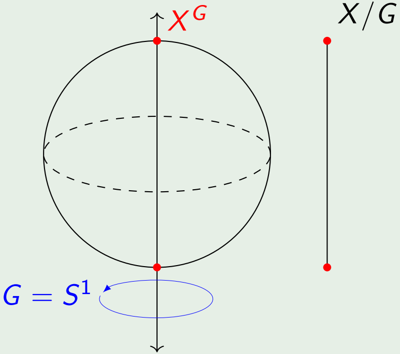

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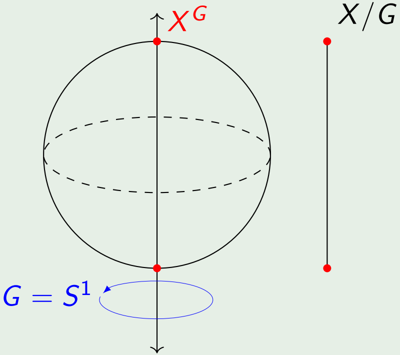

Here, $H^*(X/G) \cong H^*(pt)$.

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The orbit space $BG := EG/G$ is called *the classifying space of G*.

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Example

- $G = S^1$, $EG = S^\infty$, $BG = \mathbb{C}P^\infty$.
- $G = \mathbb{Z}/2$, $EG = S^\infty$, $BG = \mathbb{R}P^\infty$.
- $G = \mathbb{Z}$, $EG = \mathbb{R}$, $BG = S^1$.

Equivariant Cohomology

Definition (Seminar on transformation groups - A. Borel. 1960.)

For a G-space X, the Borel construction of X is the space $X_G = (EG \times X)/G$ and the G-equivariant cohomology of X is defined as

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Example

Let $X = pt$ be the "single-point" space. Then $X_G = (EG \times pt)/G = EG/G \cong BG$. Therefore,

$$H^*_G(pt) = H^*(BG)$$

If $G = S^1$, then $H^*_G(pt) = H^*(\mathbb{C}P^\infty)$ is a polynomial ring in one variable.

Particular group actions

- If G acts on X trivially (i.e. $X^G = X$) we have

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In particular for any G-space X, G acts trivially on the fixed point subspace and then

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- If G acts on X freely, we have

$$X_G \simeq X/G \text{ and } H^*_G(X) \cong H^*(X/G).$$

Equivariant Formality

For $z \in EG$, the inclusion of the fiber $i_z : X \to X_G$ given by $i_z(x) = [z, x]$ induces a map

$$r : H^*_G(X) \to H^*(X).$$

Proposition

Suppose that the map $r\colon H^*_G(X) \to H^*(X)$ is surjective. Then there is an isomorphism

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"This is a consequence of the *Leray-Hirsch Theorem*". In this case, we say that X is G-equivariantly formal.

A module structure on $H^*_G(X)$

For any G-spaces X, Y and a G-equivariant map $f : X \to Y$ (i.e. $f(g \cdot x) = g \cdot f(x)$), there is an induced map

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Remark

The map $f_G^* \colon H_G^*(Y) \to H_G^*(X)$ is a map of $H^*(BG)$-modules.

Free modules

Remember:

Let M be a module over a ring R. We say that M is a *free* module, if there is an isomorphism $M \cong R^k$ for some $k \geq 1$.

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Summary

1. The map $r\colon H^*_G(X) \to H^*(X)$ is surjective (X is G-equivariantly formal).
2. There is an isomorphism $H^*_G(X) \cong H^*(BG) \otimes H^*(X)$.
3. $H^*_G(X)$ is a free $H^*(BG)$-module.

We know that (1) $\Rightarrow$ (2) $\Rightarrow$ (3), but they are equivalent under extra assumptions (e.g. G connected).

Comparing subgroups

Let $K \subseteq G$ be a subgroup. Then any G-space X becomes a K-space by restriction of the action.

If X is G-equivariantly formal then it is K-equivariantly formal.

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Theorem [Allday - Hauschild - Puppe (2002)]

Let $G = S^1 \times \cdots S^1 = (S^1)^n$ and let X be a G-space. X is G-equivariantly formal if and only if it is K-equivariantly formal for any subgroup $K \cong S^1$.

Looking at torus actions

- A torus T is a group homeomorphic to $(S^1)^n$ for some $n \geq 1$.

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- A torus T is a group homeomorphic to $(S^1)^n$ for some $n \geq 1$.
- $H^*(BT) = H^*((\mathbb{C}P^\infty)^n) \cong \Bbbk[t_1, \ldots, t_n]$ where $\deg(t_i) = 2$.
- Let G be a compact connected Lie group. It admits a maximal torus subgroup $T \subseteq G$

Assume $\Bbbk$ is a field of characteristic 0.

Theorem [Hsiang (1975)]

Let X be a G-space where G is a compact connected Lie group. Denote by T a maximal torus of G. Then

- $H_T^*(X) \cong H_G^*(X) \otimes_{H^*(BG)} H^*(BT)$.
- X is G-equivariantly formal if and only if it is T-equivariantly formal.

The Betti number criterion

For a topological space X, denote its Betti sum by

$$b(X) = \sum_{i \geq 0} \dim_k H^i(X).$$

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In particular, if the action of T on X is free, then $X^T = \emptyset$ and then $H_T^*(X)$ is NOT equivariantly formal.

Examples

Example

Recall S^2 with the rotation action of S^1. It is S^1-equivariantly formal as $b(S^2) = b(S^0) = 2$,

$$H^*_{S^1}(S^2) \cong H^*(BS^1) \otimes H^*(S^2).$$

The same idea applies to the action of S^1 on $S^3 \subseteq \mathbb{C}^2$ given by $z \cdot (u, v) = (zu, v)$. Here $(S^3)^{S^1} \cong S^1$ and

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Example

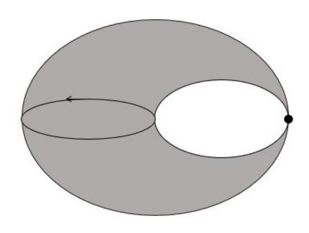
The Hopf fibration $S^1 \to S^3 \to S^2$ arises from a free action of S^1 on S^3 (This can be described as $\mathbb{C}P^1 \cong S^3/S^1$). Then

$$H^*_{S^1}(S^3) \cong H^*(S^2)$$

A non-equivarianlty formal example

Example

Let X be the space obtained from S^2 by identifying the north and south poles.



X is homotopy equivalent to a wedge of a sphere and a circle. Thus $b_0(X) = 1, b_1(X) = 1, b_2(X) = 1$ and so $b(X) = 3$. On the other hand, X^{S^1} consists of a single fixed point, then $b(X) = 1$. Therefore, X is not equivariantly formal.

Involutions as group actions

Let $\tau\colon X \to X$ be an involution. Then X has an induced action of the group $G = \{id, \tau\} \cong \mathbb{Z}/2$. Conversely, any action of $\mathbb{Z}/2$ on X give rise to an involution on X.

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Definition

A 2-torus is a group $G \cong (\mathbb{Z}/2)^n$ for some $n \geq 0$. If G acts on a space X, this is equivalent to X having n-commuting involutions.

Let $\Bbbk$ be a field of characteristic 2 now.

Remark

For any 2-torus G, we have that

$$H^*(BG) = H^*((\mathbb{R}P^\infty)^n) \cong \Bbbk[w_1, \ldots, w_n]$$

where $\deg(w_i) = 1$.

Comparing with the torus case

Betti sum criterion (Borel -1960)

Let G be a 2-torus and X a G-space with $b(X) < \infty$. X is G-equivariantly formal if and only if $b(X) = b(X^G)$.

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Equivariant formality is not captured by subgroups

There is a space X with an action of $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ such that X is not equivariantly formal, but it is with respect to every subgroup $K \subseteq G$, $K \cong \mathbb{Z}/2$.

Remark

Let $T = (S^1)^n$ be a torus. Then the subgroup G consisting of those elements $g \in T$ such that $g^2 = e$, is a 2-torus isomorphic to $(\mathbb{Z}/2)^n$.

Comparing with the torus case

Theorem [S. - Franz (2018)]

Let X be a space with an action of a torus T. Let $G \subseteq T$ be its 2-torus subgroup. Then

- $H_G^*(X) \cong H_T^*(X) \otimes_{H^*(BT)} H^*(BG)$.
- X is T-equivariantly formal if and only if it is G-equivariantly formal.

Coefficients Matter

Let X be the "Croissant" space with the action of S^1. Then $X^G \cong S^1 \vee S^1$ and thus $b(X^G) = b(X)$. This implies that X is T-equivariantly formal.

Using group cohomology

Remark

As G is a finite group, $H^*(BG) \cong H^*_{grp}(G)$. In particular, for a $\mathbb{k}[G]$-module M.

$$H^*(BG; M) \cong H^*_{grp}(G; M)$$

Theorem

Let X be a space with an action of a 2-torus G. Then,

- $H^*_G(X) \cong H^*_{grp}(G; C^*(X))$.
- *There is a homotopy equivalence*

$$C^*(X_G) \simeq Tot(H^*_{grp}(G) \otimes C^*(X))$$

- *X is G-equivariantly formal if and only if $H^*_G(X)$ is a free module over $H^*(BG)$.*

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