Equivariant cohomology for c-symplectic spaces

Sergio Chaves

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Following [Borel, 1960], let G be topological group, $EG \to BG$ a universal principal bundle for G and let X be a topological space with a continuous action of G, or a *G-space*. The equivariant cohomology of X, denoted by $H^*_G(X; R)$, is the cohomology of $H^*(X_G; R)$ where $X_G = (X \times EG)/G$ is the Borel construction of X. This object inherits a canonical structure as a module over $H^*(BG; R)$. We say that X is G-equivariantly formal if $H^*_G(X; R)$ is a free module over $H^*(BG; R)$.

1 Equivariant cohomology for the real locus of symplectic manifolds

The G-equivariant cohomology of a G-space X is closely related to the topology of its fixed point set X^G. This situation has appeared in more specific contexts such as the cohomology of compact symplectic manifolds; in fact, following Atiyah [Atiyah, 1982, Thm. 1], and extending Frankel's results in Kähler manifolds [Frankel, 1959, §4]. we cite the following theorem.

Theorem 1.1. *Let M be a compact symplectic manifold with a Hamiltonian action of a torus T. Then there is an additive isomorphism*

$$H^*(M; k) \cong \bigoplus_{i=1}^{m} H^{*-d_i}(F_i; k)$$

where F_i, $i = 1, \dots, n$ are the connected components of M^T, d_i is the Bott-Morse index of F_i; that is, d_i is the number of negative eigenvalues of the Hessian matrix associated to the critical submanifold F_i under the Morse-Bott function $f = \|\mu\|^2$. Here μ denotes the moment map associated to the torus action.

This isomorphism is actually extended to the case of T-equivariant cohomology; namely,

$$H^*_T(M; \mathbb{R}) \cong \bigoplus_{i=1}^{N} H_T^{*-d_i}(F_i; \mathbb{R}) \tag{1.1.1}$$

as shown by Kirwan in [Kirwan, 1984, §5] following Atiyah-Bott [Atiyah and Bott, 1984, Thm. 3.5]. In particular, Theorem 1.1 implies that the Betti sum of M and M^T are the same and it follows that M is T-equivariantly formal over $\mathbb{R}$.

Motivated by the case where M is a complex projective space and the complex conjugation $\tau : M \to M$ is an anti-symplectic involution (i.e. $\tau^*\omega = -\omega$, where ω denotes the symplectic form of M) and compatible with the torus action, Duistermaat [Duistermaat, 1983, Thm. 3.1] proved an analogous version of Theorem 1.1 for the fixed point subspace M^τ, commonly known as the *Real Locus of M*.

Theorem 1.2. *Let (M,ω) be a symplectic manifold with a Hamiltonian action of a torus T and a compatible anti-symplectic involution τ. There is an additive isomorphism*

$$H^*(M^\tau;\mathbb{F}_2) = \bigoplus_{i=1}^{N} H^{*-\frac{d_i}{2}}(F_i^\tau;\mathbb{F}_2)$$

and $b(M^\tau) = b(M^\tau \cap M^T)$, where $M^T = \bigcup_{i=1}^{m} F_i$.

Furthermore, in [Biss et al., 2004, Thm. A], an equivariant version of Theorem 1.2 was proved by Biss-Guillemin-Holm. Explicitly, the action of T on M induces an action of the subgroup $T_2 = \{g \in T : g^2 = 1\}$ on M^τ and the equivariant cohomology satisfies,

$$H^*_{T_2}(M^\tau;\mathbb{F}_2) \cong \bigoplus_{i=1}^{N} H^{*-\frac{d_i}{2}}_{T_2}(F_i^\tau;\mathbb{F}_2) \tag{1.2.1}$$

as $H^*(BT_2;\mathbb{F}_2)$-modules. They also showed that $b(M^\tau) = b(M^\tau \cap M^{T_2}) = b((M^\tau)^{T_2})$. In particular, this implies that M^τ is T_2-eqivariantly formal over $\mathbb{F}_2$.

Remark 1.3. When M is a symplectic manifold with a Hamiltonian action of a torus T and a compatible anti-symplectic involution τ, similar to Chapter 4, we have an induced action of $G = T \rtimes \mathbb{Z}/2$ and $M^G = (M^\tau)^T = M^\tau \cap M^T$.

Now we are interested in relating the T-equivariant cohomology of M with the T_2-equivariant cohomology of M^τ. First, it can be shown that a symplectic manifold M with an action of a torus T is equivariantly formal if and only if the T-action is Hamiltonian (see Corollary 2.7 below); therefore, combining Theorems 1.1, 1.2 and 1.2.1 we can state the following theorem.

Theorem 1.4. *Let M be a symplectic manifold with an action of a torus T and a compatible involution τ. If M is T-equivariantly formal over $\mathbb{R}$, then the real locus M^τ is T_2-equivariantly formal over $\mathbb{F}_2$.*

If M is a complex projective space, we have that M satisfies Theorem 1.4 and also $b(M)$ and $b(M^\tau)$ have the same Betti sum; this implies that M is also τ-equivariantly formal. However, the next example exhibits a symplectic manifold M where Theorem 1.4 applies, but M is not τ-equivariantly formal.

Example 1.5. Consider the symplectic manifold (S^2,ω) where $\omega \in H^2(S^2)$ is a generator. Let $T = S^1$ act on S^2 as the rotation along the z-axis, set $(M,\gamma) = (S^2 \times S^2, p_1^*\omega - p_2^*\omega)$ where $p_i \colon M \to S^2$ is the projection onto the i-th factor. Let $\tau : M \to M$ be the involution given by $\tau(x,y) = (y,x)$, then τ is an anti-symplectic involution; and consider the action of T on M given by $g\cdot(x,y) = (g\cdot x, g^{-1}\cdot y)$, the action is compatible with the involution and therefore, from Theorems 1.1 and 1.2, $H^*_T(M;\mathbb{R})$ is free over $H^*(BT;\mathbb{R})$ and $H^*_{T_2}(Y;\mathbb{F}_2)$ is free over $H^*(BT_2;\mathbb{F}_2)$ where $Y = M^\tau \cong S^2$ and T_2 is the 2-torus in T. This also follows from the Betti sum criteria; namely, M^T consist of 4-points, and thus $b(M) = b(M^T) = 4$. Also, Y^{T_2} consists of two points and thus $b(Y) = b(Y^{T_2}) = 2$; however, $b(Y) < b(M)$ and thus the equivariant cohomology $H^*_{\mathbb{Z}/2}(M;\mathbb{F}_2)$ is not free over $H^*(B\mathbb{Z}/2;\mathbb{F}_2)$ where the action of $\mathbb{Z}/2$ on M is the one given by the involution τ.

On the other hand, we immediately have a condition for M being τ-equivariantly formal.

Proposition 1.6. *Let M be a symplectic manifold with a Hamiltonian action of a torus T and a anti-symplectic involution τ. Then M is τ-equivariantly formal over $\mathbb{F}_2$ if and only if $b(M) = b(M^H)$ where H is the 2-subtorus in $G = T \rtimes \mathbb{Z}/2$. The latter acts on M via the induced action of T and τ.*

By Proposition 1.6, it is enough to assume that X is T-equivariantly formal for X^τ to be T_2-equivariantly formal in the symplectic setting. Now in the most general possible case, we have the following question.

Question 1.7. *Let X be a T-space together with a compatible involution τ. Assume that $b(X) < \infty$ and $H_T^*(X;\mathbb{F}_2)$ is a free $H^*(BT;\mathbb{F}_2)$-module. Is $H_{T_2}^*(X^\tau;\mathbb{F}_2)$ a free $H^*(BT_2;\mathbb{F}_2)$- module where the action of the 2-torus $T_2 \subseteq T$ on X^τ is the one induced by the action of T on X ?*

Without extra assumptions on the space, a negative answer can be given as we will describe in the next proposition.

Proposition 1.8. *There exists a manifold X with an action of $T = S^1$ and a compatible involution τ such that X is T-equivariantly formal and the real locus X^τ is not equivariantly formal with respect to the induced action of the 2-torus subgroup $T_2 \subseteq T$.*

Proof. let $X = \{(u,z) \in \mathbb{C}\times\mathbb{R} : |u|^2 + |z|^2 = 1\} = S^2$, let $T = S^1$ act on X by $g\cdot(u,z) = (gu,z)$; more precisely, by scalar multiplication in the first factor. Let τ be the involution $\tau(u,z) = (\bar{u},-z)$ which is compatible with the torus action. Notice that $X^T = \{(0,1),(0,-1)\} \cong S^0$ and $X^\tau = \{(-1,0),(1,0)\} \cong S^0$. Therefore, the action of T_2 on X^τ is the multiplication by ± 1 and thus it is a free T_2-space, this implies that its T_2-equivariant cohomology is not free over $H^*(BT_2)$. On the other hand, $H_T^*(X)$ is a free $H^*(BT)$-module since X and X^T have the same Betti sum. □

One of the main issues of this example is that $X^G = \emptyset$, even assuming $X^G \neq \emptyset$ a counterexample of question 1.7 can be found and its construction is motivated by [Su, 1964, Sec. 5]. First we recall the following well known construction of topological spaces.

Definition 1.9. Let $f : X \to Y$ be a G-map between G-spaces X and Y. The mapping cylinder is defined as the G-space $M_f = (X\times[0,1]) \sqcup Y/\sim$ where $(x,1) \sim f(x)$, with the action given by $g\cdot(x,t) = (gx,t)$ for $(x,t) \in X\times[0,1]$ and the regular action on Y; notice that it is well defined at the points of the form $(x,1)$ since f is a G-map.

The space M_f is G-homotopic to Y, and therefore $H^*(M_f) \cong H^*(Y)$. Also, the fixed point subspace $(M_f)^G \cong M_{f^G}$ where $f^G : X^G \to Y^G$. Now let $g : X \to Z$ be a G-map and M_g the respective mapping cylinder, then the space $M_{f,g} = M_f \cup_{X\times\{0\}} M_g$ has cohomology groups fitting in the long exact sequence

$$0 \to H^0(M_{f,g}) \to H^0(Y)\oplus H^0(Z) \to H^0(X) \to H^1(M_{f,g}) \to \cdots$$

following from the Mayer-Vietoris long exact sequence. Moreover, $M_{f,g}$ becomes a G-space and $(M_{f,g})^G \cong M_{f^G,g^G}$. In particular, we have

Proposition 1.10. *Let m,n,r be different integers, $h : S^m \to S^n$ a map between spheres and consider $f = h\times id : S^m\times S^r \to S^n\times S^r$ and $g : S^m\times S^r \to S^m$ the projection. Then $H^*(M_{f,g})$ is free over $\mathbb{Z}/2$ where a copy of $\mathbb{Z}/2$ happens in degree $0, n, m+r+1, n+r$ and it is zero otherwise. In particular, $b(M_{f,g}) = 4$.*

Example 1.11. Let $X = S^3 \subseteq \mathbb{C}^2$, $Y = S^5 \subseteq \mathbb{C}^3$ and $Z = S^9 \subseteq \mathbb{C}^4$. Let $T = S^1$ act on X and Y by scalar multiplication on the first component , respectively, and let T act on Z by scalar multiplication on the first and second component and trivially otherwise. Then $X^T = S^1$, $Y^T = S^3$ and $Z^T = S^5$. Let τ act on X and Y as the complex conjugation on the first component respectively, and on Z as the complex conjugation on the first and second component and multiplication by -1 on the other components. Then $X^\tau = S^2$, $Y^\tau = S^4$ and $Z^\tau = S^1$. Note that the induced action of $T_2 \subseteq T$ is free on Z^τ.

Let $f : X\times Z \to Y\times Z$ be the map $i\times id$ where i is the inclusion $i(u,z) = (u,z,0)$, and $g : X\times Z \to X$ the projection. Consider the T-space $M = M_{f,g}$ and the induced action of τ on M becomes a compatible involution. Then $b(M) = b(M^T) = b(M^\tau) = 4$ from Proposition 1.10, but $b(M^G) = b((M^\tau)^{T_2}) = 2$.

Example 1.12. Let $X = S^3$, $Y = S^2$ and $h : X \to Y$ be the Hopf map. This map can be explicitly presented as $h(u,z) = (2u\bar{z}, |u|^2 - |z|^2)$ where S^3 is seen as the unit sphere in $\mathbb{C}^2$ and S^2 as the unit sphere in $\mathbb{C} \times \mathbb{R}$. Let $T = S^1$ act on S^3 and S^2 as the complex multiplication in the first component respectively, and τ be the involution on S^3 and S^2 given by the complex conjugation in the first component respectively. Then τ is compatible with the torus action and $X^T \cong S^1$, $X^\tau \cong S^2$, $Y^T \cong S^0$ and $Y^\tau \cong S^1$. Now let $Z = S^5$ be the unit sphere in $\mathbb{C}^3$, let T act on Z by multiplication in the first component and τ be the involution on Z given by the complex conjugation in the first component, and multiplication by -1 in the second and third component; then $Z^T \cong S^3$ and $Z^\tau \cong S^0$, notice that the action of the 2-torus $T_2 \subseteq T$ on Z^τ is free.

Let $M = M_{f,g}$ be the construction of Proposition 1.10 , then $b(M) = b(M^T) = b(M^\tau) = 4$ and thus M is T-equivariantly formal; nevertheless, M^τ is not T_2-equivariantly formal since $b((M^\tau)^{T_2}) = 2 < b(M^\tau)$.

These examples provide a negative solution for Question 1.7 with non-empty common fixed points of both T and τ. Summarizing we get.

Proposition 1.13. *There is a topological space M with an action of a torus T and a compatible involution τ such that $M^G \neq \emptyset$, M is T-equivariantly formal and $\mathbb{Z}/2$-equivariantly formal, but the real locus M^τ is not T_2-equivariantly formal with respect to the induced action of the 2-torus $T_2 \subseteq T$ on M^τ.* □

2 Cohomologically symplectic spaces

Definition 2.1. Let M be a $\Bbbk$-Poincare duality space with formal dimension $2n$. We say that M is a c-symplectic space (cohomologically symplectic), if there is a class $\omega \in H^2(M;\Bbbk)$ such that $\omega^n \neq 0$.

Notice that any compact symplectic manifold is a c-symplectic space; however, there exists c-symplectic spaces which do not admit a symplectic structure. For instance, the connected sum $\mathbb{C}P^2\#\mathbb{C}P^2$ is c-symplectic but it does not admit a symplectic form [Audin, 1991, Prop 1.3.1].

Definition 2.2. A *c*-symplectic space M satisfies the weak Lefschetz condition if the multiplication by ω^{n-1} induces an isomorphism $H^1(M;\Bbbk) \cong H^{2n-1}(M;\Bbbk)$. Moreover, if the multiplication by ω^r induces an isomorphism $H^{n-r}(M;\Bbbk) \cong H^{n+r}(M;\Bbbk)$ for $r = 1, \dots, n-1$. we say that M satisfies the strong Lefschetz condition. Since any Kähler manifold satisfies the strong Lefschetz condition, we define analogously the cohomological version. More precisely, a c-symplectic space satisfying the strong Lefschetz condition is called a c-Kähler (cohomologically-Kähler) space.

An important property of the c-Kähler spaces is a condition over its Betti numbers, as we remark in the following result.

Proposition 2.3. *Let M be a c-Kähler space, then the odd Betti numbers $b_{2k+1}(M)$ are even.*

Proof. Let $s = 2k + 1$. From a standard result in symplectic linear algebra, it is enough to show that $H^s(M)$ admits a non-degenerate skew-symmetric bilinear form. Let $1 \leq r \leq n-1$ be such that $s = n + r$ or $s = n - r$, assume without loss of generality the latter. Consider the isomorphism $\phi : H^s(M) \to H^{2n-s}(M)$ given by multiplication by ω^r from the strong Lefschetz condition. Consider the non degenerate pairing given by Poincaré duality $\mu : H^s(M) \times H^{2n-s}(M) \to \Bbbk$. Then the bilinear form on $H^s(M)$ given by $\Omega(a,b) = \mu(a, \phi^{-1}(b))$ is a non-degenerate skew-symmetric form and thus $H^s(M)$ is an even-dimensional vector space. □

Definition 2.4. Let G be a torus if $\Bbbk = \mathbb{Q}$ or a p-torus if $\Bbbk = \mathbb{F}_p$. An action of G on M is said to be c-Hamiltonian (cohomologically Hamiltonian) if $\omega \in \operatorname{Im}(i^* \colon H^*_G(M;\Bbbk) \to H^*(M;\Bbbk))$.

Proposition 2.5. *Let M be a c-symplectic space with an action of a torus T. Assume that M satisfies the weak Lefschetz condition and that $M^T \neq \emptyset$, then the action is c-Hamiltonian.*

Proof. We can assume that T is a circle. Consider the spectral sequence associated to the fibration $M \to M_T \to BT$, write $d_2(\omega) = x \cdot c \in H^1(M) \otimes H^2(BT)$ where $c \in H^2(BT)$ is a generator. Since M is a Poincare duality space and $M^T \neq \emptyset$, $\omega^n \in \mathrm{Im}(H^*_T(M) \to H^*(M))$ and thus $0 = d_2(\omega^n) = n\omega^{n-1}x \cdot c$. From the weak Lefschetz condition we get that $x = 0$ and so $d_2(\omega) = 0$. Since this is the only possible non-zero differential on ω, we get $\omega \in \mathrm{Im}(H^*_T(M) \to H^*(M))$. □

In the case of symplectic manifolds, the Hamiltonian torus action and c-Hamiltonian torus action are the same. This can be shown using the Cartan model for equivariant cohomology. [Mukherjee, 2005, Prop. 1.5.6]

Theorem 2.6. *Let M be a symplectic manifold with an action of a torus T. The action is Hamiltonian if and only if it is c-Hamiltonian.*

From Theorem 1.1 we get that a Hamiltonian torus action is equivariantly formal. On the other hand, if M is T-equivariantly formal, the map $H^*_T(M) \to H^*(M)$ is surjective and thus the action is c-Hamiltonian. Therefore, from 2.6 we obtain the following result.

Corollary 2.7. *Let M be a symplectic manifold with an action of a torus T. The action is Hamiltonian if and only if M is T-equivariantly formal.*

The following lemma, whose proof requires standard results in algebraic top logy, will allow us to construct further examples.

Lemma 2.8.

(a) Let M and N be connected $\mathbb{k}$-orientable manifolds of the same dimension n. Then $b_i(M\#N) \cong b_i(M) + b_i(N)$ for $i \neq 0, n$ and $b_0(M) = b_n(M) = 1$.

(b) Let M be a connected manifold of dimension $n \geq 2$. Denote by $\widetilde{M}$ the manifold obtained by "attaching a handle" on M; more precisely, remove two open sets $U, V \cong D^n$ of M and then glue a cylinder $S^{n-1} \times I$ along the common boundary $S^{n-1} \sqcup S^{n-1}$. Then , $b_i(\widetilde{M}) = b_i(\widetilde{M})$ for $i \neq n-1, 1$; moreover, when $n > 2$ we have that $b_j(\widetilde{M}) = b_j(M) + 1$ for $j = 1, n-1$ and if $n = 2$ we have $b_1(\widetilde{M}) = b_1(M) + 2$.

Proof. Let C denote the gluing cylinder in both cases. To prove (a), observe that collapsing C into a point we get an isomorphism $H^*(M\#N, C) \cong H^*((M\#N)/C, C) \cong \widetilde{H}^*(M \vee N)$. Using the cohomology long exact sequence for the pair $(M\#N, C)$ and that $H^*(C) = H^*(S^{n-1})$ we have immediately that $\widetilde{H}^i(M\#N) \cong \widetilde{H}^i(M \vee N)$ for $i \neq n, n-1$. To compute the remaining degrees, we look at the short exact sequence

$$0 \to H^{n-1}(M \vee N) \to H^{n-1}(M\#N) \to H^{n-1}(S^{n-1}) \to H^n(M \vee N) \to H^n(M\#N) \to 0$$

where the map $H^n(M \vee N) \to H^n(M\#N)$ coincides with the surjective map $\mathbb{k} \otimes \mathbb{k} \to \mathbb{k}$ as M, N and $M\#N$ are $\mathbb{k}$-orientable and thus $H^{n-1}(M \vee N) \cong H^{n-1}(M\#N)$. To prove (b) we use the Mayer-Vietoris long exact sequence. We write $\widetilde{M} = N \cup S$ where N is homeomorphic to M with two discs removed U and V, and $S \cong S^{n-1}$; also we have that the intersection $N \cap S \cong S^{n-1} \sqcup S^{n-1}$. Therefore, the Mayer-Vietoris long exact sequence yields

$$0 \to \mathbb{k} \to H^1(\widetilde{M}) \to H^1(N) \oplus H^1(S^{n-1}) \to H^1(S^{n-1}) \oplus H^1(S^{n-1}) \to \cdots$$

$$\cdots \to H^{n-1}(\widetilde{M}) \to H^{n-1}(N) \oplus H^{n-1}(S^{n-1}) \to H^{n-1}(S^{n-1}) \oplus H^{n-1}(S^{n-1})$$

$$\to H^n(\widetilde{M}) \to H^n(N) \to 0$$

Therefore, $b_i(\widetilde{M}) = b_i(N)$ for $i \neq 1$, $b_1(\widetilde{M}) = b_1(N) + 1$ and $b_n(\widetilde{M}) = 1$. It only remains to compute the Betti numbers of N. To do so, we use again the Mayer-Vietoris sequence for the decomposition $M = N \cup W$ where $W = U \cup V \cong D^n \sqcup D^n$. In this case, the sequence give us $b_i(N) = b_i(M)$ for $i \neq n-1, n$, $b_{n-1}(N) = b_{n-1}(M)+1$ and $b_n(N) = 0$. The statement of the lemma follows by combining the results from the two sequences discussed above. □

If M is a c-symplectic space which is T-equivariantly formal, then the action is automatically c-Hamiltonian. However, if M admits a c-Hamiltonian action of a torus T, it is not necessarily T-equivariantly formal; as we state in the following proposition.

Proposition 2.9 (C. Allday [Allday, 1998]). *There exist a c-symplectic space M satisfying the weak Lefschetz condition together with a c-Hamiltonian action of a circle T and $b(M^T) < b(M)$. Thus M is not T-equivariantly formal.*

Proof. Let $T = S^1$ act freely on $X = S^3 \times S^3$. By the equivariant tubular neighborhood consider a tube $U = S^1 \times D^5$ around an orbit where T acts by multiplication on the first factor. Remove the interior of the tube and glue $D^2 \times S^4$, where T acts by rotations on the first factor. Call the resulting manifold N, then N is a T-space where $N^T = S^4$. Using the Mayer-Vietoris long exact sequence, we obtain that the Betti numbers of $H^*(N;\mathbb{Q})$ are $1,0,1,2,1,0,1$ in degree $0,1,\ldots,6$ respectively; in fact, let $N_0 = X \setminus U$ which is homeomorphic to $S^3 \times S^3$ with an orbit removed around a chosen point. Let $V \cong S^1 \times D^5$ be such that $X = N_0 \cup V$ and $N_0 \cap V \cong S^1 \times (D^5 \setminus \{0\})$ which has the homotopy type of $S^1 \times S^4$. Applying the Mayer-Vietoris long exact sequence for such decomposition we get that $H^*(N_0)$ has dimension $1,0,0,2,1,0,0$ in degree $0,1,\ldots,6$ respectively. Now, N is constructed by gluing N_0 and $D^2 \times S^4$ along the common boundary $S^1 \times S^4$, this provides a decomposition of N into open sets $U \subseteq V$ such that $U \cong N_0$, $V \cong S^4$, $U \cup V = N$ and $U \cap V \cong S^1 \times S^4$. From the Mayer-Vietoris long exact sequence the Betti numbers of $H^*(N;\mathbb{Q})$ are the ones stated above.

Now let T act on $\mathbb{C}P^3$ by $g\cdot[z_0 : z_1 : z_2 : z_3] = [gz_0 : z_1 : z_2 : z_3]$, so the fixed point subspace is homeomorphic to $p \sqcup \mathbb{C}P^2$ where $p = [1 : 0 : 0 : 0]$. Let $M = N\#\mathbb{C}P^3$ be the equivariant connected sum formed by removing T-invariant discs around fixed points of $p \in N^T = S^4 \subseteq N$ and $y \in (\mathbb{C}P^3)^T = \mathbb{C}P^2 \subseteq \mathbb{C}P^3$; the existence of the T-invariant discs follows from the Equivariant tubular neighborhood theorem (Theorem **??**). This implies then that $b(M) = 8$. Notice that $M^G \cong p \cup (S^4\#\mathbb{C}P^2) \cong p \cup \mathbb{C}P^2$; moreover, the non-trivial form $\omega \in H^2(\mathbb{C}P^3;\mathbb{Q})$ induces a non trivial form $\tilde{\omega} \in H^2(M;\mathbb{Q})$ such that $\tilde{\omega}^3 \neq 0$, so M is a c-symplectic space. Moreover, since $\mathbb{C}P^3$ is T-equivariantly formal, the lifting of the class ω to $H_T^*(\mathbb{C}P^3;\mathbb{Q})$ induces a lifting of $\tilde{\omega}$ to $H_T^*(M;\mathbb{Q})$ and thus the action of T on M is c-Hamiltonian. However, since $b(M^T) = 4 < b(M) = 8$, M is not T-equivariantly formal and therefore Theorem 1.1 does not hold in the case of c-symplectic spaces. □

Even though we may consider formal Hamiltonian torus action on c-symplectic spaces, Question 1.7 in the setting of c-symplectic spaces does not have a positive solution. The example will be constructed in a similar fashion as in Proposition 2.9.

Proposition 2.10. *There exists a c-symplectic space M with a torus action of T and a compatible c-antisymplectic involution τ such that M is T-equivariantly formal over $\mathbb{Q}$ and the real locus M^τ is not equivariantly formal over $\mathbb{F}_2$ with respect to the induced action of the 2-torus subgroup $T_2 \subseteq T$.*

Proof. Let $T = S^1$, consider the T-action on $S^3 \subseteq \mathbb{C}\times\mathbb{C}$ given by $g\cdot(u,z) = (gu, z)$ and the involution τ defined as $\tau(u,z) = (\bar{u}, -z)$, then τ is compatible with the T-action and we have an induced action of $G = T \rtimes \mathbb{Z}/2$;

moreover, $(S^3)^T \cong S^1 \subseteq \{0\} \times \mathbb{C}$ and $(S^3)^\tau \cong S^0 \subseteq \mathbb{C} \times \{0\}$. Let $X = S^3 \times S^3$ be the G-space with the induced diagonal action, let $Y = \mathbb{C}P^3$ be the G-space with the action given by $g \cdot [z_0 : z_1 : z_2 : z_3] = [gz_0 : gz_1 : z_2 : z_3]$ for $g \in T$ and the involution τ defined as the complex conjugation, then $X^T \cong S^1 \times S^1$, $X^\tau \cong S^0 \times S^0$, $Y^T \cong \mathbb{C}P^1 \sqcup \mathbb{C}P^1$, $Y^\tau \cong \mathbb{R}P^3$. Notice that the induced $T_2 \cong \mathbb{Z}/2$-action on X^τ is free while $(Y^\tau)^{T_2} \cong \mathbb{R}P^1 \sqcup \mathbb{R}P^1$.

Choose points $x \in X^T \setminus X^\tau$ and $y \in Y^T \setminus Y^\tau$, then the orbit spaces $G \cdot x \cong G \cdot y$ consist of two points and the stabilizers $G_x \cong G_y \cong T$. By the equivariant tubular neighborhood theorem (Theorem **??**), there exist $U \subseteq X$, $V \subseteq Y$ G-invariant neighborhoods of x and y respectively such that $U \cong V \cong D^6 \times \mathbb{Z}/2$, T acts by scalar multiplication on the complex components of $D^6 \subseteq (\mathbb{C} \times \mathbb{R})^2$ and τ is the complex conjugation on D^6 on the complex components, and multiplication by -1 on the real components and the $\mathbb{Z}/2$ factor. Let M be the space obtained as "a double connected sum" by removing U and V from X and Y respectively and gluing the spaces $X \setminus U$ and $Y \setminus V$ along a double cylinder $I \times S^5 \times \mathbb{Z}/2$ where G acts trivially on the unit interval I and on $S^5 \times \mathbb{Z}/2$ as the restriction of the action on the boundary of $D^6 \times \mathbb{Z}/2$ described above. Note that M can be obtained by attaching a handle to the connected sum $X \# Y$. Therefore, from Lemma 2.8, the Betti numbers of M are $1, 1, 1, 2, 1, 1, 1$ in degree $0, 1, \ldots, 6$ respectively and thus $b(M) = 8$. We have also that $M^T \cong M_0 \sqcup \mathbb{C}P^1$ where M_0 is homeomorphic to a "double connected sum" between $S^1 \times S^1$ and $\mathbb{C}P^1$, which is indeed homeomorphic to a genus 2 surface. Therefore, $b(M^T) = b(M_0) + b(\mathbb{C}P^1) = 6 + 2 = 8$ so M is a T equivariantly formal space. Moreover, the symplectic form $\Omega \in H^2(\mathbb{C}P^3)$ induces a c-symplectic form $\omega \in H^2(M)$ which admits an equivariant lifting $\tilde{\omega} \in H^2_T(M)$ and thus the action of T on M is c-Hamiltonian. On the other hand, $M^\tau \cong (S^0 \times S^0) \sqcup \mathbb{R}P^3$ and thus $b(M^\tau) = 4 + 4 = 8$; however, $(M^\tau)^{T_2} \cong \mathbb{R}P^1 \sqcup \mathbb{R}P^1$ and $b((M^\tau)^{T_2}) = 4$, that is, M^τ is not equivariantly formal with respect to the T_2-action. ◻

This example is a c-symplectic space which does not satisfy the weak Lefschetz condition, that is, the multiplication by $\omega^2 : H^1(M) \to H^5(M)$ is clearly zero. Indeed, since $H^3(M)$ is generated by the elements $a, b \in H^3(S^3 \times S^3)$ and in the cohomology of the connected sum $H^*(S^3 \times S^3 \# \mathbb{C}P^3)$ we have that $a \cdot \omega = b \cdot \omega = 0$, the same equation holds in the cohomology $H^*(M)$. This implies that for the generator $x \in H^1(M)$ we have $x \cdot \omega = \lambda_1 a + \lambda_2 b$ for some $\lambda_1, \lambda_2 \in \{0, 1\}$ and thus $x \cdot \omega^2 = 0$.

In the case of c-Kähler spaces, any torus action with non empty fixed points is formal and thus c-Hamiltonian. This follows from this stronger result due to A. Blanchard [Blanchard, 1956, Thm. II.1.2].

Theorem 2.11. *Let X be c-Kähler space (over a field $\Bbbk$) and $X \to E \to B$ be a fiber bundle. Consider cohomology with coefficients over a field $\Bbbk$. Suppose that $\pi_1(B)$ acts trivially in the cohomology $H^*(F)$, then the Serre spectral sequence collapses and*

$$H^*(E) \cong H^*(B) \otimes H^*(X).$$

As an immediate result, for any connected group K acting on X, if the fixed point subspace $X^K \neq \emptyset$, X is K-equivariantly formal. Now we can prove Duistermaat's theorem in the case of c-Kähler spaces.

Proposition 2.12. *Let X be a c-Kähler space (over $\Bbbk = \mathbb{F}_2$) with an action of a torus T an a compatible anti-symplectic involution τ. Assume that τ acts trivially in the cohomology of X. Then X is T-equivariantly formal over $\Bbbk$ and the real locus X^τ is T_2 equivariantly formal over $\Bbbk$.*

Proof. From Blanchard's result, we have that X is T-equivariantly formal over $\Bbbk$. This implies that it is also T_2-equivariantly formal by Corollary **??** and so T_2 acts trivially on the cohomology of X. By assumption, τ acts trivially on the cohomology of X as well and thus the group $H = T_2 \times \tau$ acts trivially in the cohomology of X. Using again Theorem 2.11 we obtain that X is H-equivariantly formal. Finally, the T_2-equivariant formality of the real locus X^τ follows from Theorem **??**. ◻

References

[Allday, 1998] Allday, C. (1998). Examples of circle actions on symplectic spaces. *Banach Center Publications*, 45(1):87–90.

[Atiyah and Bott, 1984] Atiyah, M. and Bott, R. (1984). The moment map and equivariant cohomology. *Topology*, (23):1–28.

[Atiyah, 1982] Atiyah, M. F. (1982). Convexity and commuting hamiltonians. *Bulletin of the London Mathematical Society*, 14(1):1–15.

[Audin, 1991] Audin, M. (1991). Exemples de variétés presque complexes. *l'Enseignement Math.*, 37:175–190.

[Biss et al., 2004] Biss, D., Guillemin, V. W., and Holm, T. S. (2004). The mod 2 cohomology of fixed point sets of anti-symplectic involutions. *Advances in Mathematics*, 185(2):370–399.

[Blanchard, 1956] Blanchard, A. (1956). Sur les variétés analytiques complexes. In *Annales scientifiques de l'École Normale Supérieure*, volume 73, pages 157–202. Elsevier.

[Borel, 1960] Borel, A. (1960). Seminar on transformation groups, with contributions by g. bredon, ee floyd, d montgomery, r palais. *Annals of Mathematics Studies*, 46.

[Duistermaat, 1983] Duistermaat, J. (1983). Convexity and tightness for restrictions of hamiltonian functions to fixed point sets of an antisymplectic involution. *Transactions of the American Mathematical Society*, 275(1):417–429.

[Frankel, 1959] Frankel, T. (1959). Fixed points and torsion on kähler manifolds. *Annals of Mathematics*, pages 1–8.

[Kirwan, 1984] Kirwan, F. C. (1984). *Cohomology of quotients in symplectic and algebraic geometry*, volume 31. Princeton University Press.

[Mukherjee, 2005] Mukherjee, G. (2005). *Transformation groups: symplectic torus actions and toric manifolds*. Springer.

[Su, 1964] Su, J. (1964). Periodic transformations on the product of two spheres. *Transactions of the American Mathematical Society*, 112(3):369–380.