Cobordism and Groups of Homotopy Spheres

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Introduction

At the development of the algebraic topology in the $20th$ century, mathematicians were interested in relating homotopy and homology invariants to topology and smoothness of spaces. In 1900, Henry Poincaré, studying the topological properties of the standard 3-sphere, claimed that a manifold with the same homology as S^3 must be actually homeomorphic to the sphere. Later, he answered his own question by constructing a homology sphere with non-zero fundamental group. This is a classical example in algebraic topology and it is known as the *Poincaré Sphere*. So, he modified the claim by conjecturing that a compact, oriented and connected manifold space with the same homotopy as S^3 is a space homeomorphic to the sphere. This problem was known as the *Poincaré Conjecture*, and remained open for more than a century.

Poincaré's statement was generalized to higher dimensions, known as the *Generalized Poincaré Conjecture:* A compact, connected and oriented topological space with the homotopy type as S^n (or a *Homotopy Sphere*), is homeomorphic to the n-dimensional euclidean sphere. Note that by the *Hurewicz Theorem*, a space is a homotopy sphere if and only if it is a homology sphere and simply connected. Moreover, since it was believed that the sphere had an unique smooth structure, the Conjecture turned into the statement that a Homotopy sphere is diffeomorphic to S^n (the n-dimensional sphere with the standard smooth structure).

So, the "exotic" differentiable structures over S^n were presumed non-existent until 1956, when J. Milnor [M1] constructed a 7-dimensional manifold homeomorphic to the sphere which is non-diffeomorphic to S^7, yielding to a complete theory about the study of this kind of manifolds known as *Exotic Spheres.*

The study of the Generalized Poincaré Conjecture and exotic spheres have in common the tools provided by *Cobordism Theory*, which allows to distinguish between manifolds homeomorphic but non-diffeomorphic to the sphere. The first remarkable result is the Thom–Pontryagin theorem, where the cobordism and the homotopy are intrinsically related, and this theorem gives rise to corollary results as the characterization of the stable homotopy groups $\pi_{n+k}(S^k)$ as framed cobordism and the *Hirzebruch Signature Theorem.*

With this theory at hand, in 1952, S. Smale proved the the following result:

The h-cobordism Theorem
Let M,N be smooth and simply connected n-dimensional manifolds with $n \geq 5$. If W is a compact h-cobordism between M and N, then W is diffeomorphic to $M \times [0,1]$.

The Generalized Poincaré Conjecture for the case $n \geq 5$ follows immediately from this result. Later, in 1982, M. Freedman provided a proof for the case $n = 4$. Finally, G. Perelman solved the Poincaré's Conjecture in 2006.

The h-cobordism theorem, which allows to characterize the topology of the spheres by its homotopy type, also provides tools for distinguishing the exotic smooth structures over the spheres. M. Kervaire and J. Milnor consider the class of topological n-dimensional spheres under the equivalence relation of h-cobordism, which coincides with the class of topological spheres under the diffeomorphism relation. This set has a group structure given by the connected sum operation, it is a finite group and it is explicitly computable for many values of n.

In other words, *the number of n-dimensional exotic spheres is finite* (up to diffeomorphism).

The above results are condensed in *Homotopy Groups of Spheres I* [KM], but many results and proofs are left for a second paper *Homotopy Groups of Spheres II* which never was published. Although the problem of different smooth structures over S^n is almost completely solved, [1] the main aim of this work is to complete the lacking content in [KM] and provide an step-by-step study of the exotic spheres theory.

So, in Chapter 2 the generalities of the cobordism theory, the proof of the Thom–Pontryagin theorem and the Hirzebruch Signature Theorem are presented . Later in Chapter 3, different examples of exotic spheres in suitable dimensions are constructed. Finally the Chapter 4 are focused in determining completely the groups of exotic spheres, finishing with the explicit computation of the order of this groups in dimensions less than 30 presented in Chapter 5.

[1]The number of exotic spheres in dimension 4 is still unknown

Contents

Chapter 1

Preliminaries

1.1 Vector Bundles and Characteristic Classes

This section is made for setting notation and main results related to fundamental theory of vector bundles and characteristic classes, which will be used throughout the whole work. For a complete treatment in this topics refer to [MS].

Definition 1.1.1. Let $K = \mathbb{R}$ or $\mathbb{C}$. *A real (or complex) vector bundle of rank n over X*, is a triple $\xi : K^n \to E \xrightarrow{\pi} X$, where E and X are topological spaces, and $\pi : E \to X$ is a continuous map; which satisfies the following conditions

(i) For any $p \in X$ the fibre $F_p = \pi^{-1}(p)$ of π over p is a vector space isomorphic to K.

(ii) Every $p \in X$ has a neighbourhood U, such that there is a homeomorphism

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times K^n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle pr_1} \\ U & \xrightarrow{id_U} & U \end{array}$$

and the diagram commutes, which means that every fibre F_p is mapped to $\{p\} \times K$.

(iii) $\phi_U \restriction_{F_p} : F_p \to K$ is an isomorphism of vector spaces.

Proposition 1.1.2. Let X be a topological space. The following are examples of vector bundles over X.

1. The trivial bundle $\epsilon^n : K^n \to K^n \times X \to X$.

2. If $\eta : K^n \to F \xrightarrow{\pi} Y$ is a vector bundle and $f : X \to Y$ is a continuous function, the pullback of η is the bundle $f^*\eta : K^n \to E \to X$, where E is the set of all pairs $(b, e) \in X \times F$ with $f(b) = \pi(e)$

3. If $X = M$ a smooth n-dimensional manifold, the tangent bundle $\mathbb{R}^n \to TM \to M$.

4. If $X = M$ a smooth n-dimensional manifold and $i : M \hookrightarrow \mathbb{R}^{n+k}$ is an embedding, the normal bundle $\nu(i) : \mathbb{R}^k \to E \to M$ where the space $E \subseteq M \times \mathbb{R}^n$ is the set of all pairs (x, v) such that v is orthogonal to the tangent space T_xM.

□

Proposition 1.1.3. Let $f, g : X \to Y$ be continuous maps and ξ a vector bundle over Y. If f and g are homotopic, then $f^*\xi \cong g^*\xi$. □

Definition 1.1.4. Let ξ_1, ξ_2 two vector bundles over the same base space B. Let $d : B \to B \times B$ denote the diagonal embedding. The bundle $d^*(\xi_1 \times \xi_2)$ over B is called the *Whitney sum* of ξ_1 and ξ_2, and it will be denoted by $\xi_1 \oplus \xi_2$, since each fiber $E_b(\xi_1 \oplus \xi_2)$ is canonically isomorphic to the direct sum $E_b(\xi_1) \oplus E_b(\xi_2)$.

Example 1.1.5. Let M be a smooth manifold and $i : M \hookrightarrow \mathbb{R}^N$ an embedding. We have that $TM \oplus \nu_M(i) \cong \mathbb{R}^N \times M$, that is, the Whitney sum of the normal bundle and the tangent bundle of a manifold is trivial.

Definition 1.1.6. A *framing* φ of a n-dimensional vector bundle ξ is an isomorphism $\varphi : \xi \cong \epsilon^n$. In this case, we say that the bundle ξ is *trivializable* with a choice of trivialization.

Denote by t^n the standard framing of ϵ^n.

Definition 1.1.7. Let M be a smooth manifold. M is *parallelizable* if the bundle TM is trivializable. M is said *stably parallelizable* if the bundle $TM \oplus \epsilon^1$ is trivializable.

Recall that the Euclidean sphere S^n is stably parallelizable.

Theorem 1.1.8. Let M be a n-dimensional smooth manifold with boundary. M is stably parallelizable if and only if M is parallelizable. □

It is known that the set of all n-dimensional planes through the origin of the space $\mathbb{R}^{n+k}$ denoted by $Gr_n(\mathbb{R}^{n+k})$, is a compact manifold of dimension nk. It is called the *Grasmann manifold*. Recall that for the case $n = 1$, $Gr_1(\mathbb{R}^{k+1})$ is equal to the real projective space $\mathbb{R}P^k$.

There is a canonical vector bundle over $Gr_n(\mathbb{R}^{n+k})$ denoted by $\gamma^n(\mathbb{R}^{n+k})$. Let E be the set of all pairs (X, v) such that X is a n-plane in $\mathbb{R}^{n+k}$ and $v \in X$. The projection map $E \to Gr_n(\mathbb{R}^k)$ is defined by $\pi(X, v) = X$. The fiber E_X over X, is canonically isomorphic to X.

Let $\mathbb{R}^\infty$ denote the vector space consisting of those sequences $x = (x_1, x_2, \ldots)$ of real numbers for which all but a finite number of the x_i are zero. For a fixed k, the subspace consisting of all $x = (x_1, \ldots, x_k, 0, \ldots)$ will be identified with the coordinate space $\mathbb{R}^k$. Thus $\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots$ with union $\mathbb{R}^\infty$.

Definition 1.1.9. The *infinite Grassmann manifold* $BO(n) = Gr_n(\mathbb{R}^\infty)$ is the set of all n-dimensional linear subspaces of $\mathbb{R}^\infty$, topologized as the direct limit of the sequence

$$Gr_n(\mathbb{R}^n) \hookrightarrow Gr_n(\mathbb{R}^{n+1}) \hookrightarrow Gr_n(\mathbb{R}^{n+2}) \hookrightarrow \cdots$$

A canonical bundle γ^n over $BO(n)$ is constructed just as in the finite dimensional case.

Theorem 1.1.10. Let ξ be an n-plane bundle over a paracompact base B. Then there exists a map $B \xrightarrow{f_\xi} BO(n)$ such that $f^*\gamma^n \cong \xi$. Furthermore, if $\xi \cong \eta$, then the maps f_ξ, f_η are homotopic. □

Corollary 1.1.11. For any topological space X, the set of n-dimensional real vector bundles of over X (up to isomorphism) is in a bijective correspondence with the set of homotopy class of maps $X \to BO(n)$. □

In the above notation, f_ξ is the *classifying map* of the bundle ξ. Sometimes the classifying map of a bundle will be denoted by the bundle itself, that is, for a bundle ξ, the notation for its classifying map is $X \xrightarrow{\xi} BO(n)$.

Definition 1.1.12. The *Stiefel-Whitney* classes satisfy the following axioms:

(A1) To each vector bundle ξ there corresponds a sequence of cohomology classes

$$w_i(\xi) \in H^i(X, \mathbb{Z}_2),$$

for $i = 0, 1, \cdots$. The class $w_0(\xi)$ is equal to the unit element

$$1 \in H^0(X, \mathbb{Z}_2)$$

and $w_i(\xi)$ equals zero for i greater than n if ξ is an n-plane bundle.

(A2) If $f : B(\xi) \to B(\eta)$ is a map such that $f^*\eta \cong \xi$, then

$$w_i(\xi) = f^* w_i(\eta).$$

(A3) If ξ and η are vector bundles over the same base space, then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta).$$

(A4) For the line bundle $\gamma^1(\mathbb{R}P^1)$, the Stiefel-Whitney class $w_i(\gamma^1(\mathbb{R}P^1))$ is non zero.

Definition 1.1.13. The *Thom Space* of a real vector bundle ξ over a compact space X, denoted by $T\xi$ is defined to be the one-point compactification of the space E, this point will be denoted by $t_0(\xi)$. We write $MO(k)$, for the Thom space $T\gamma^k$. Suppose there is a vector bundle η over a space C and a continuous map $g : B \to C$. If we let $\xi = g^*\eta$, then g induces a map $Tg : (T\xi, t_0(\xi)) \to (T\eta, t_0(\eta))$.

Theorem 1.1.14. There is an isomorphism $\widetilde{H}^{n+k}(T\xi, \mathbb{Z}_2) \cong H^k(B, \mathbb{Z}_2)$ for any n-dimensional vector bundle ξ. □

Definition 1.1.15. Let M be a n-dimensional smooth manifold and let N be a closed smooth manifold. Let $g : M \to (T\xi - t_0)$ be a smooth map, where ξ is a k-dimensional bundle over N. Then g is said to be *transverse* at the inclusion given by the zero-section $N \hookrightarrow T\xi$ if for all $x \in g^{-1}(N)$

$$Im(dg_x) + T_{g(x)}N = T_{g(x)}(T\xi - t_0).$$

In particular, $g^{-1}(N) \subseteq M$ is a closed $(n-k)$-dimensional submanifold.

Theorem 1.1.16 (Transversality Theorem)**.** With the above notation, every smooth map $M \to (T\xi - t_0)$ is homotopic to a map $g : M \to T\xi$ which is transverse at the zero section. □

An orientation on a real vector space V of dimension n is a choice of an equivalence class of ordered bases, where two bases are equivalent if there exists a linear transformation with positive determinant which send a basis onto the other one. Then, there are two possible orientations on a real vector space.

This is equivalent to a choice of a generator $\mu_V \in H_n(V, V_0, \mathbb{Z}) \cong \mathbb{Z}$ where $V_0 = V - 0$. It gives rise to a generator $u_V \in H^n(V, V_0, \mathbb{Z}) \cong \mathbb{Z}$ by the relation $\langle u_V, \mu_V \rangle = 1$.

Definition 1.1.17. An *orientation* on a n-plane bundle ξ, is a choice of orientation on each fiber satisfying the usual local triviality condition: For every point $b \in B(\xi)$, there exist a local coordinate system (N, h) with $b \in N$ and $h : N \times \mathbb{R}^N \to \pi^{-1}(N)$ with each fiber F_c over N, the homomorphism $x \mapsto h(x, c)$ from $\mathbb{R}^N$ to F_c is orientation preserving.

For any n-plane bundle ξ, let E_0 be the set of all non-zero elements of E and let F_0 be the set of all non-zero elements of a fiber $F = \pi^{-1}(b)$.

Theorem 1.1.18 (Oriented Thom Isomorphism theorem)**.** Let ξ be an oriented n-plane bundle with total space E. Then the cohomology group $H^i(E, E_0, \mathbb{Z})$ is zero for $i < n$ and $H^n(E, E_0, \mathbb{Z})$ contains an unique cohomology class u_ξ (the Thom class) whose restriction

$$u_\xi|_{(F,F_0)} \in H^n(F, F_0, \mathbb{Z}),$$

is equal to the generator u_F for every fiber F of ξ. Furthermore, the correspondence $x \mapsto x \cup u$ defines an isomorphism from $H^k(E, \mathbb{Z})$ to $H^{k+n}(E, E_0, \mathbb{Z})$ for every integer k. □

For an oriented n-plane bundle $\xi = (E, B, \pi)$, the *oriented Thom isomorphism* $\varphi : H^i(B, \mathbb{Z}) \to H^{i+n}(E, E_0, \mathbb{Z})$ is defined by $\varphi(x) = \pi^*(x) \cup u_\xi$.

Definition 1.1.19. Let ξ be an oriented n-plane bundle over B. The *Euler class* $e(\xi) \in H^n(B, \mathbb{Z})$ is defined by

$$e(\xi) = \varphi^{-1}(u_\xi \cup u_\xi).$$

That is, $\pi^* e(\xi) = u_\xi|_E$

Remark. The Euler class has many similar properties of Stiefel Whitney classes. Some of them are listed here.

1. If $f : B(\xi) \to B(\eta)$ is covered by an orientation preserving bundle map form ξ to η, then
$$e(\xi) = f^* e(\eta).$$
2. If ξ and η are oriented vector bundles over the same base space, then
$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta).$$
3. If the orientation of a oriented vector bundle ξ es reversed, then the Euler class changes sign.
4. If ξ is an oriented n-plane bundle with n odd, then $2e(\xi) = 0$.
5. Let ξ be an oriented vector bundle over the base space B. The canonical map $H^n(B, \mathbb{Z}) \to H^n(B, \mathbb{Z}_2)$ sends $e(\xi)$ to $w_n(\xi)$.

A final application of the Euler class is the construction of an oriented Gysin sequence.

Proposition 1.1.20 (Oriented Gysin Sequence)**.** Let ξ be an oriented n-plane bundle with projection $\pi : E \to B$. Let $\pi_0 : E_0 \to B$ be the restriction of π to E_0. Then for any coefficient ring R, there is a long exact sequence

$$\cdots \to H^i(B, R) \xrightarrow{\cup e} H^{i+n}(B, R) \xrightarrow{\pi_0^*} H^{i+n}(E_0, R) \to$$

where e denotes the image of $e(\xi)$ in $H^n(B, R)$ under the homomorphism of cohomology induced by the ring map $\mathbb{Z} \to R$. □

Remark. Note that given any complex n-plane bundle ω, we can forget the complex structure and consider each fiber as a real vector space of dimension $2n$. Thus we obtain the underlying real $2n$-plane bundle $\omega_{\mathbb{R}}$. If $v_1, \cdots, v_n$ is a $\mathbb{C}$-basis for a fiber F of ω, we take $v_1, iv_1, \cdots, v_n, iv_n$ to be an $\mathbb{R}$-basis for F as a fiber of $\omega_{\mathbb{R}}$. So we have the following result.
For a complex n-plane bundle ω with total space E, we define a $(n-1)$-plane bundle ω_0 over E_0 as follows. Given a pair $(b, v) \in E_0$, $v \in F_b$, $v \neq 0$, let the fiber over (b, v) in ω_0 be the orthogonal complement of v in F_b.

Proposition 1.1.21. If ω is a complex vector bundle, then the underlying real vector bundle $\omega_{\mathbb{R}}$ has a canonical preferred orientation. □

Definition 1.1.22. Let ω be a complex n-plane bundle over B. For $i \leq n$, the i^{th} *Chern class* $c_i(\omega) \in H^{2i}(B, \mathbb{Z})$ is defined inductively as follows: Set $c_n(\omega) = e(\omega_{\mathbb{R}})$. For $i < n$. Set $c_i(\omega) = \pi_0^{-1} c_i(\omega_0)$. For $i > n$, set $c_i(\omega) = 0$.

The Chern classes satisfy the following properties

1. If $f : B(\omega) \to B(\omega')$ is covered by a bundle map form ω to ω', then
$$c(\omega') = f^* c(\omega).$$
2. If ω and ω' are complex vector bundles over the same base space, then
$$c(\omega \oplus \omega') = c(\omega) c(\omega').$$
3. The conjugate bundle $\overline{\omega}$ has Chern classes
$$c_i(\overline{\omega}) = (-1)^i c_i(\omega).$$

Let ξ be a real n-plane bundle. The *complexification* of ξ is defined by the complex n-plane bundle with the same base space and fiber $F \otimes_{\mathbb{R}} \mathbb{C}$, where F denotes a fiber of ξ.

It is clear that $(\xi \otimes_{\mathbb{R}} \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$, and also we have $\xi \otimes_{\mathbb{R}} \mathbb{C} \cong \overline{\xi \otimes_{\mathbb{R}} \mathbb{C}}$, since the conjugation is a $\mathbb{R}$-linear homomorphism. Therefore, since $c_i(\overline{\xi \otimes_{\mathbb{R}} \mathbb{C}}) = (-1)^i c_i(\xi \otimes_{\mathbb{R}} \mathbb{C})$, the odd Chern classes $c_1, c_3, \cdots$ of the complexification of a real vector bundle are 2-torsion elements.

Definition 1.1.23. Let ξ be an n-plane bundle over B. The i^{th} *Pontryagin class* $p_i(\xi) \in H^{4i}(B, \mathbb{Z})$ is defined by
$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes_{\mathbb{R}} \mathbb{C}).$$
The *total Pontryagin class* $p(\xi) \in H^*(B, \mathbb{Z})$ is the cohomology class
$$p(\xi) = p_1(\xi) + \cdots + p_{[n/2]}(\xi).$$
The properties of Pontryagin classes follows from those of Chern classes. We list some of them here.

1. If $f : B(\xi) \to B(\eta)$ is covered by a bundle map from ξ to η, then
$$p(\xi) = f^* p(\eta).$$

2. If ξ and η are vector bundles over the same base space, then

$$2p(\xi \oplus \eta) = 2p(\xi)p(\eta).$$

There is a relation between the Pontryagin classes with the Euler class.

Proposition 1.1.24. If ξ is an oriented $2n$-plane bundle, then $p_n(\xi) = e(\xi)^2$. □

Theorem 1.1.25. Let R be an integral domain containing $\frac{1}{2}$. Let p_i the image of $p_i(\widetilde{\gamma}^n)$ and e the image of $e(\widetilde{\gamma}^n)$ under the cohomology map induced by the ring map $\mathbb{Z} \to R$. Then for odd n,

$$H^*(BSO(n), R) \cong R[p_1, \ldots, p_{\frac{n-1}{2}}],$$

and for even n,

$$H^*(BSO(n), R) \cong R[p_1, \ldots, p_{\frac{n-2}{2}}, e].$$

Corollary 1.1.26. $H^i(BSO(n), \mathbb{Z})$ if finite if i is not divisible by 4 and has rank $p(i/4)$ if i is divisible by 4.

Given any partition I of n, there exist an unique polynomial $s_I \in \mathbb{Z}[t_1, \ldots, t_n]$ satisfying

$$s_I(\sigma_1, \cdots, \sigma_n) = \sum t^I = \sum t_1^{r_1} \cdots t_k^{r_k},$$

where the $\sum$ indicates that we take every monomial that can be formed with exponents exactly $(r_1, \cdots, r_k)$. And $\sigma_1, \cdots, \sigma_n$ are the elementary symmetric functions of the ring $\mathbb{Z}[t_1, \ldots, t_n]$.

Definition 1.1.27. Let M be a compact oriented manifold of dimension $4n$. There is a fundamental homology class $\mu_M \in H_{4n}(M, \mathbb{Z})$. For any vector bundle ξ over M and any partition $I = (i_1, \ldots, i_k)$ of n, we define a *Pontryagin number*

$$P_I[\xi] = \langle p_{i_1}(\xi) \cdots p_{i_k}(\xi), \mu_M \rangle \in \mathbb{Z},$$

and a s-number

$$S_I[p(\xi)] = \langle s_I(p_1(\xi), \ldots, p_n(\xi)), \mu_M \rangle \in \mathbb{Z}.$$

Proposition 1.1.28. Let ξ and η be vector bundles over M. Then

$$2s_I(p(\xi \otimes \eta)) = 2 \sum_{I_1 I_2 = I} s_{I_1}(p(\xi)) s_{I_2}(p(\eta)).$$

If ξ' is a vector bundle over another manifold N, then

$$S_I[p(\xi \times \eta)] = \sum_{I_1 I_2 = I} S_{I_1}[p(\xi)] S_{I_2}[p(\eta)].$$

□

Example 1.1.29. Let $\tau = T\mathbb{C}P^n$, $\tau_{\mathbb{R}} \otimes \mathbb{C} \cong \tau \oplus \overline{\tau}$, thus

$$\begin{aligned} c(\tau_{\mathbb{R}} \otimes \mathbb{C}) &= c(\tau \oplus \overline{\tau}) \\ 1 + c_2(\tau_{\mathbb{R}} \otimes \mathbb{C}) + \cdots + c_{2n}(\tau_{\mathbb{R}} \otimes \mathbb{C}) &= c(\tau)c(\overline{\tau}) \\ 1 - p_1(\tau_{\mathbb{R}}) + p_2(\tau_{\mathbb{R}}) - \cdots \pm p_n(\tau_{\mathbb{R}}) &= (1-x)^{n+1}(1+x)^{n+1} \\ p(\tau_{\mathbb{R}}) &= (1+x^2)^{n+1} \end{aligned}$$

Therefore

$$P_{(n)}[\tau_{\mathbb{R}}] = n + 1$$

Let $\widetilde{Gr}_n(\mathbb{R}^{n+k})$ denote the Grassmann manifold consisting of all oriented n-planes in $\mathbb{R}^{n+k}$. This can be topologized in order to give a manifold structure of dimension nk. $\widetilde{Gr}_n(\mathbb{R}^{n+k})$ is a double covering of the unoriented Grassmann manifold $Gr_n(\mathbb{R}^{n+k})$. Passing to the direct limit $k \to \infty$, we obtain the *oriented infinite Grassmann manifold*

$$BSO(n) = \lim_{k\to\infty} \widetilde{Gr}_n(\mathbb{R}^{n+k}).$$

The covering map $f_n : BSO(n) \to BO(n)$ lifts to an oriented n-plane bundle over $BSO(n)$, $f_n^*\gamma^n = \widetilde{\gamma}^n$.

Proposition 1.1.30. Let B be a topological space. There is a bijective correspondence between the homotopy class $[B, BSO(n)]$ and the set of oriented n-plane bundles over B (up to isomorphism).

Definition 1.1.31. Let $K = \mathbb{R}$ or $\mathbb{C}$. The *Stiefel Manifold* $St_{n,k}(K)$ is the set consisting of k-tuples $v = (v_1, \ldots, v_k)$ of orthonormal vectors in K^n, under the equivalence relation $v \sim w$ if and only if $Span(v) = Span(w)$.

Proposition 1.1.32. There are diffeomorphism $St_{n,k}(\mathbb{R}) \cong O(n)/O(n-k)$ and $St_{n,k}(\mathbb{C}) \cong U(n)/U(n-k)$. Furthermore, $St_{n,k}(\mathbb{R})$ is $(n-k-1)$ connected.

Theorem 1.1.33. Let B a CW-complex of dimension n and let ξ be an n-dimensional real vector bundle over B. There exists a framing of ξ over the j-skeleton of B if and only if a certain well defined obstruction class

$$\sigma_j(\xi) \in H^j(B, \pi_{j-1}(St_{n,n-j+1}(\mathbb{R})))$$

is zero.

1.2 Homotopy Groups

Let G_n denote one of the following groups: $O(n)$, $SO(n)$ or $U(n)$. There are natural inclusions $G_n \overset{i_n}{\hookrightarrow} G_{n+1}$ and define the direct limit associated to these sequence as

$$G = \lim_{n\to\infty} G_n$$

The following theorem is due to Bott [Bt]

Theorem 1.2.1 (Bott Periodicity Theorem).

1. $\pi_*(U)$ is periodic with period 2, $\pi_0(U) = 0$, $\pi_1(U) = \mathbb{Z}$.

2. $\pi_*(O)$ is periodic with period 8 and the homotopy groups are

n mod 8	0	1	2	3	4	5	6	7
$\pi_n(O)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

3. For all n, there are isomorphism

$$\pi_n(U/SO) \cong \pi_{n-2}(SO)$$

Moreover, Borel-Hirzebruch [BH] prove that

$$\pi_{2n}(U(n)) \cong \mathbb{Z}_{n!}$$

for $n > 0$. □

Together with the *Freundenthal Suspension Theorem*, another important result about homotopy groups of spheres was proved by Serre. [Sr]

Theorem 1.2.2. For $k \geq n+2$, the homotopy group $\pi_{n+k}(S^k)$ is independent of k. Moreover, the group $\pi_{n+k}(S^k)$ is finite. □

This group is denoted the n^{th}-*stable homotopy group of spheres*, Π_n.

Chapter 2

Thom–Pontryagin Theorem

In his Ph.D. thesis [Th], René Thom, relates cobordism theory with stable homotopy theory, and since the Thom–Pontryagin theorem was initially intended as an approach to the computation of homotopy groups of spheres, the application to yield information about manifolds shows that it is highly productive.

2.1 Cobordism Categories

Definition 2.1.1. A *cobordism category* $(\mathcal{C}, \partial, i)$ is a triple satisfying:

1. $\mathcal{C}$ a category having finite sums (coproducts) and an initial object $\emptyset$.
2. $\partial : \mathcal{C} \to \mathcal{C}$ an additive functor with $\partial\partial M = \emptyset$ for any object M and $\partial\emptyset = \emptyset$.
3. $i : \partial \to Id$ a natural transformation of additive functors, where Id denotes the identity functor.
4. $\mathcal{C}$ has a essentially small subcategory $\mathcal{C}_0$ (a Set) such that each element of $\mathcal{C}$ is isomorphic to an element of $\mathcal{C}_0$.

Example 2.1.2. In the case of the category of differentiable manifolds, where the sum operation is given by disjoint union, we take $\emptyset$ to be the empty manifold and i to be given by the inclusion of ∂M in M. The existence of a small subcategory follows from the Whitney embedding theorem.

The fundamental notion in cobordism is the following equivalence relation.

Definition 2.1.3. In a cobordism category, two objects M, N are *cobordant*, $M \equiv N$ if there exist objects V, W such that $M + \partial V \cong N + \partial W$.

This relation of cobordism has these properties.

Proposition 2.1.4.

1. $\equiv$ is an equivalence relation on $\mathcal{C}$, and the equivalence classes form a set.
2. If $M \equiv N$ then $\partial M \cong \partial N$.
3. For all M, $\partial M \equiv \emptyset$.
4. If $M \cong N$ and $M' \cong N'$ then $M + M' \cong N + N'$.

Proof.

1. Reflexivity and symmetry follow from the properties of isomorphism. For transitivity, suppose that $M \equiv N$ and $N \equiv L$, then there exist objects U, V, X, Y of $\mathcal{C}$ with $M + \partial U \cong N + \partial V$, and $N + \partial X \cong L + \partial Y$. So

$$\begin{aligned} M + \partial(U + X) &\cong M + \partial U + \partial X \\ &\cong N + \partial V + \partial X \\ &\cong L + \partial V + \partial Y \\ &\cong L + \partial(V + Y). \end{aligned}$$

   The fact that the equivalence classes form a set follows from the existence of the small subcategory $\mathcal{C}_0$.

2. If $M \equiv N$, then there exists objects X, Y with $M + \partial X \cong N + \partial Y$. Then

$$\begin{aligned} \partial M &\cong \partial M + \emptyset \\ &\cong \partial M + \partial\partial X \\ &\cong \partial(M + \partial X) \\ &\cong \partial(N + \partial Y) \\ &\cong \partial N + \partial\partial Y \\ &\cong \partial N + \emptyset \cong \partial N. \end{aligned}$$

3. $\partial M + \partial\emptyset \cong \emptyset + \partial M$, since $\partial\emptyset = \emptyset$. Therefore $\partial M \equiv \emptyset$.

4. We have $M + \partial X \cong N + \partial Y$ and $M' + \partial X' \cong N' + \partial Y'$, for some objects X, X', Y, Y'. Then we have $M + M' + \partial(X + X') \cong N + N' + \partial(Y + Y')$.

□

In the case of differentiable manifolds, the original definition of cobordism states that two manifolds without boundary M, N are cobordant, if there exist a manifold W with boundary such that $M + N \cong \partial W$. Now we show that these two definitions are equivalent.

Proposition 2.1.5. In the case of manifolds without boundary, categorical definition for cobordism agrees with the original one.

Proof. Suppose that M and N are cobordant in the categorical sense. Then there exist manifolds X, Y with $M + \partial X \cong N + \partial Y$. Let $W_1 = M \times I + X$ and $W_2 = N \times I + Y$. Since $M \equiv N$, W_1 and W_2 can be glued along that common boundary to form a manifold W with $\partial W \cong M + N$.
Conversely, now suppose that there exist a manifold W such that $\partial W \cong M + N$. Then $M + \partial W \cong M + M + N \cong N + \partial(M \times I)$. So $M \equiv N$. □

Indeed, the set of equivalence classes of the cobordism relation form a semigroup, as we will show with the following definitions and results.

Definition 2.1.6. An object M of $\mathcal{C}$ is *closed* if $\partial M \cong \emptyset$. We say M *bounds* if $M \equiv \emptyset$.

Actually, these definitions are compatible with the cobordism relation and sum operation.

Proposition 2.1.7. Let M, N objects in $\mathcal{C}$.

1. Suppose $M \equiv N$. Then M is closed if and only if N is closed, and M bounds if and only if N does.
2. If M and N are both closed, then $M + N$ is closed. If M and N both bound, then $M + N$ bounds.
3. If M bounds then M is closed.

Proof.
1. The statement about closed objects follows from the property 2 in the lemma (2.1.4), and the statement about bounding objects follows from the fact that $\equiv$ is an equivalence relation.
2. If M and N are closed, then $\partial M \cong \emptyset \cong \partial M$. Thus $\partial(M + N) \cong \partial M + \partial N \cong \emptyset + \emptyset \cong \emptyset$. From the property 4 in the lemma (2.1.4) follows that if M and N both bound, then $M + N$ also bounds.
3. If M bounds then $M \equiv \emptyset$. Thus, by property 2 of lemma (2.1.4), $\partial M \cong \partial\emptyset \cong \emptyset$. So M is closed.

□

Now immediately we have the following result.

Theorem 2.1.8. The set of equivalence classes of $\mathcal{C}$ under the relation of cobordism has a commutative, associative operation induced by the addition in $\mathcal{C}$. The class of $\emptyset$ provides an identity element for this operation. □

This allows to make this definition.

Definition 2.1.9. The *Cobordism Semigroup* $\Omega(\mathcal{C}, \partial, i)$ is the set of equivalence classes of closed objects of $\mathcal{C}$ with the operation induced by the addition in the category.

2.2 (B, f) Manifolds

In order to compute the cobordism semigroups, we need to consider manifolds endowed with additional structure.

Definition 2.2.1. Let $f_k : B_k \to BO(k)$ be a fibration [1]. Let $\xi : M \to BO(k)$ be a k-vector bundle over M. A (B_k, f_k) *structure* on ξ is an equivalence class of liftings $\widetilde{\xi} : M \to B_k$ (that is, $\xi = f_k \circ \widetilde{\xi}$). This equivalence relation is given by the homotopy relation.

To give a well defined notion of a (B_k, f_k) structure on a manifold M, we will use the Whitney Embedding theorem to produce and embedding $i : M \to \mathbb{R}^N$ and we proceed to consider (B_k, f_k) over the normal bundle $\nu_M(i)$. However, we need the following lemma to allow us make a definition independent on the embedding.

Lemma 2.2.2. For a k sufficiently large, there is a bijective correspondence between the (B_k, f_k) structures on the normal bundles $\nu_M(i_1)$ and $\nu_M(i_2)$ associated to the embeddings $i_1, i_2 : M \to \mathbb{R}^{n+k}$ where M is an n-dimensional smooth manifold.

Proof. For k sufficiently large, any two embeddings i_1, i_2 are isotopic by a map $H : M \times I \to \mathbb{R}^{n+k}$. The family of normal bundles $(H|_{M\times t})^*(T\mathbb{R}^{n+k}/TM)$ gives a homotopy of $\nu_M(i_1)$ and $\nu_M(i_2)$. Thus we get a well defined equivalence relation of the two normal bundles. The bijection follows from the homotopy lifting theorem. □

[1] A *fibration* denotes a continuous map $p : E \to B$ satisfying the homotopy lifting property

Let (B, f) denote a sequence of fibrations $f_k : B_k \to BO(k)$ together with maps $g_k : B_k \to B_{k+1}$ such that the diagram commutes.

$$\begin{array}{ccc} B_k & \xrightarrow{g_k} & B_{k+1} \\ \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k+1}} \\ BO(k) & \xrightarrow{j_k} & BO(k+1) \end{array}$$

Where $j_k : BO(k) \to BO(k+1)$ is the inclusion induced by the standard inclusions $G_k(\mathbb{R}^{n+k}) \hookrightarrow G_{k+1}(\mathbb{R}^{n+k+1})$.

Now suppose we have a (B_k, f_k) structure $\widetilde{\nu}_M(i) : M \to B_k$ on the normal bundle $\nu_M(i)$ of an embedding $i : M \to \mathbb{R}^{n+k}$. This induces a (B_{k+1}, f_{k+1}) structure on the normal bundle $\nu_M(i')$ of the embedding $i' = i \times 0 : M \to \mathbb{R}^{n+k+1}$, by setting $\widetilde{\nu}_M(i') = g_r \widetilde{\nu}_M(i)$ since

$$f_{k+1}\widetilde{\nu}_M(i') = f_{k+1}g_k\widetilde{\nu}_M(i) = j_r f_r \widetilde{\nu}_M(i) = j_r \nu_M(i) = \nu(i').$$

Definition 2.2.3. A (B, f) *structure* on a manifold M is an equivalence class of compatible (B_k, f_k) structures on the normal bundles of inclusions of M under de above construction; where the equivalence is given by agreement for sufficiently large k subject to the bijection of lemma (2.2.2).

We illustrate this definition by considering the following important examples.

Example 2.2.4.

1. Let $B_k = BO(k)$ and f_k be the identity map. Every manifold will have an unique (BO, Id) structure, thus the class of (BO, Id) manifolds is simply the class of all manifolds. This class is denoted by Ω^{un}.

2. Take $B_k = BSO(k)$ and f_k the map which ignores the orientation. Every oriented manifold have an unique (BSO, f) structure because the choice of the lifting is given by the orientation. Then the class of (BSO, f) manifolds is the same class as the class of oriented manifolds. This class is denoted by Ω^{SO}.

3. Consider the fibration $O(k) \to EO(k) \xrightarrow{f_k} BO(k)$ where $EO(k)$ is a contractible manifold. A manifold M have a (B, f) structure if and only if there exists a framing of the bundle $\nu(i)$ for some embedding $i : M \to \mathbb{R}^{n+k}$. This class of (B, f) manifolds is the same class of normally framed manifold. This class is denoted by Ω^{fr}.

2.3 (B, f) Cobordism

Definition 2.3.1. Let $\mathcal{C}$ be the category whose objects are compact manifolds together with a specified (B, f) structure, and whose maps are the smooth, boundary preserving inclusions with trivial normal bundle inducing compatible (B, f) structures. Let $\partial : \mathcal{C} \to \mathcal{C}$ be the boundary functor, inducing (B, f) structures by the inner trivialization. Let $i : \partial \to I$ be the inclusion of the boundary with inner trivialization. Then $(\mathcal{C}, \partial, i)$ is a cobordism category, called *the cobordism category of* (B, f) *manifolds.*

We denote by $\Omega(B, f)$ the semigroup $\Omega(\mathcal{C}, \partial, i)$. It can be written as

$$\Omega(B, f) = \bigoplus_{n=0}^{\infty} \Omega_n(B, f),$$

where $\Omega_n(B, f)$ denotes the subsemigroup of equivalence classes of n dimensional manifolds.

In fact, $\Omega(B, f)$ is not simply a semigroup.

Proposition 2.3.2. $\Omega(B, f)$ is an abelian group.

Proof. Take a (B, f) manifold $M^n \in \Omega(B, f)$, choose an embedding $i : M \to \mathbb{R}^{n+k}$ with a lifting $\widetilde{\nu}(i) : M \to B_k$ inducing the correct (B, f) structure on M. Let $j : M \times I \hookrightarrow \mathbb{R}^{n+k+1}$ be the obvious embedding. If $\pi : M \times I \to M$ is the projection. Since $f_k \widetilde{\nu}(i)\pi = \nu(i)\pi = \nu(j)$ we get a (B, f) structure $\widetilde{\nu}(j) : M \times I \to B_k$ given by $\widetilde{\nu}(j) = \widetilde{\nu}(i)\pi$. The induced (B, f) structure on $M \times 0$ is the same as that on M, so $M \cong M \times 0$ as (B, f) manifolds. If we let $M' = M \times 1$ with the inner induced (B, f) structure, we have that $M + M' \cong \partial(M \times I) \equiv \emptyset$, and thus $M + M' \equiv \emptyset$. Hence M has an inverse, M', and $\Omega(B, f)$ is an abelian group. ☐

Apply this construction to (B, f) manifolds. There is a map $j_k : BO(k) \hookrightarrow BO(k+1)$, and we see that $j_k^*(\gamma^{k+1}) = \gamma^k \oplus \epsilon^1$, where ϵ^1 is the trivial line bundle over $BO(k)$. Note that $T(\gamma^k \oplus \epsilon^1) = \Sigma T\gamma^k$.
So we have a map $Tj_k : \Sigma T\gamma^k \to T\gamma^{k+1}$. Also we have a map $g_k^* f_{k+1}^* \gamma^{k+1} \to f_{k+1}^*$ induced by g_r. By commutativity $g_k^* f_{k+1}^* = f_k^* j_k^* \gamma^{k+1}$. Thus there is a map $f_k^* j_k * \gamma^{k+1} \to f_{k+1}^* \gamma^{k+1}$, and this yields a map

$$Tg_k : Tf_k^* j_k * \gamma^{k+1} \to Tf_{k+1}^* \gamma^{k+1}.$$

Finally, using that by definition $Tf_k^* j_k^* \gamma^{k+1} = TB_{k+1}$ and the above observations, we have that

$$\begin{aligned} Tf_k^* j_k^* \gamma^{k+1} &= Tf_k^*(\gamma^k \oplus \epsilon^1) \\ &= T(f_k^* \gamma^k \oplus f_k^* \epsilon^1) \\ &= \Sigma T f_k^* \gamma^k \\ &= \Sigma TB_k. \end{aligned}$$

So,

$$Tg_k : \Sigma TB_k \to TB_{k+1}.$$

And we obtain a new commutative diagram

$$\begin{array}{ccc} \Sigma TB_k & \xrightarrow{Tg_k} & TB_{k+1} \\ \downarrow{\scriptstyle \Sigma Tf_k} & & \downarrow{\scriptstyle Tf_{k+1}} \\ \Sigma TBO(k) & \xrightarrow{Tj_k} & BO(k+1) \end{array}$$

Since $\Sigma_\# : \pi_{n+k}(TB_k, t_0) \to \pi_{n+k+1}(\Sigma TB_k, t_0)$ and $Tg_{k\#} : \pi_{n+k+1}(\Sigma TB_k, t_0) \to \pi_{n+k+1}(TB_{k+1}, t_0)$

we obtain a map $Tg_{k\#} \circ \Sigma_{\#} : \pi_{n+k}(TB_k, t_0) \to \pi_{n+k+1}(TB_{k+1}, t_0)$.

This allows to define the homotopy group

$$\lim_{k\to\infty} \pi_{n+k}(TB_k, t_0).$$

Now finally we are in condition to state and prove the Thom–Pontryagin Theorem.

2.4 Thom–Pontryagin Theorem

Theorem 2.4.1. The cobordism group of n-dimensional (B, f) manifolds $\Omega_n(B, f)$ is isomorphic to the homotopy group $\lim_{k\to\infty} \pi_{n+k}(TB_k, t_0)$.

The proof will be focused in several steps.

Let M^n be a (B, f) manifold. Let $i : M \hookrightarrow \mathbb{R}^{n+k}$ be an embedding and $\nu = \nu(i) : M \to BO(k)$ the normal bundle associated, N the total space of this bundle and $\pi : N \to M$ the projection. Choose a lifting $\widetilde{\nu} : M \to B_k$ giving the right (B, f) structure.

Recall $N = \{(x, v) \in M \times \mathbb{R}^{n+k} | v \in T_x M^{\perp}\}$ so it can be considered as a embedded submanifold of $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$ (using the embedding i), and there is an exponential map

$$exp : \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \to \mathbb{R}^{n+k},$$

given by $exp(i(x), v) = i(x) + v$. We have that $exp|_{i(M)\times 0} = i$ and it is a differentiable map, so for some $\epsilon > 0$, $exp|_{N_\epsilon}$ is an embedding where N_ϵ is the subset of N consisting of vectors of length less or equal to ϵ.

Define $c_0 : \mathbb{R}^{n+k} \to N_\epsilon/\partial N_\epsilon$ by sending de interior of N_ϵ to itself, and $\mathbb{R}^{n+k} - int(N_\epsilon)$ to the point ∂N_ϵ. This map can be extended to the compactification of $\mathbb{R}^{n+k}$ by sending ∞ to ∂N_ϵ to obtain a map $c : S^{n+k} \to N_\epsilon/\partial N_\epsilon$.

Note that with $\epsilon = 1$, $N_1/\partial N_1 = TN$, so let $\epsilon^{-1} : N_\epsilon/\partial N_\epsilon \to TN$ be the multiplication by $1/\epsilon$. Consider $\varrho_\epsilon = \epsilon^{-1} \circ c : S^{n+k} \to TN$, this map sends $int(N_\epsilon)$ diffeomorphically to $TN - t_0$.

Now let $j_n^k : \gamma^k(\mathbb{R}^{n+k}) \to \gamma^k$ be the standard inclusion and $n : N \to \gamma^k(\mathbb{R}^{n+k})$ the bundle map $(x, v) \mapsto (N_x, v)$. Then there is a map $(j_n^k \circ n) \times (\widetilde{\nu} \circ \pi) : N \to \gamma^k \times B_k$. This is injective since n is.

If we denote by $p : \gamma^k \to BO(k)$, we have that

$$\begin{aligned} f_r(\widetilde{\nu} \circ \pi(x, v)) &= \nu(x) \\ &= p \circ j_n^k(N_x, v) \\ &= p(j_n^k \circ n(m, x)). \end{aligned}$$

So the image of this map is inside of $f_k^*\gamma^k$. Thus there is a bundle map $l : (j_n^k \circ n) \times (\widetilde{\nu} \circ \pi) : N \to f_k^*\gamma^k$, inducing a map $Tl : TN \to Tf_k^*\gamma^k = TB_k$.

Finally define $\theta_{i,\widetilde{\nu},\epsilon}(M) : (S^{n+k}, \infty) \to (TB_k, t_0)$ to be the composition $Tl \circ \varrho_\epsilon$. Observe that this map embeds $int(N_\epsilon)$ into TB_k and the rest of S^{n+k} into t_0.

For a decreasing ϵ', $\theta_{i,\widetilde{\nu},\epsilon}$ is homotopic to $\theta_{i,\widetilde{\nu},\epsilon'}$ since N_ϵ and N'_ϵ are; and for an equivalent choice of $\widetilde{\nu}$ also will give homotopic maps by the definition of equivalence of liftings. This gives a well defined $\theta_i(M) \in \pi_{n+k}(TB_k, t_0)$.

Now we will prove that actually $\theta_i(M)$ is compatible with the structure of direct limit of $\lim\limits_{k\to\infty} \pi_{n+k}(TB_k, t_0)$.

Lemma 2.4.2. Let $t : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k+1}$ the inclusion. Then the embedding $t \circ i : M \to \mathbb{R}^{n+k+1}$ gives rise to the map $Tg_k \circ \Sigma\theta_i$, that is, $\theta_{ti} = Tg_k \circ \Sigma\theta_i$.

Proof. Let $l_n^k : \gamma^k(\mathbb{R}^{n+k}) \to \gamma^{k+1}(\mathbb{R}^{n+k+1})$, $\eta : N \to E(\nu(ji))$ and $s_{n+r} : S^{n+k} \to S^{n+k+1}$ be the respective extensions of j. So

$$\widetilde{\nu}(ji) = g_k\widetilde{\nu}, \epsilon_{ji}^{-1}c_{ji}s_{n+k} = T\eta \circ \epsilon^{-1}c, \pi = \pi_{ji}\eta, n_{ji}\eta = l_n^k n$$

thus,

$$\begin{aligned}
\theta_{ji}s_{n+k} &= T((j_n^{k+1}n_{ji}) \times (\widetilde{\nu}(ji)\pi_{ji}))\epsilon_{ji}^{-1}c_{ji}s_{n+k} \\
&= T((j_n^{k+1}n_{ji}) \times (\widetilde{\nu}(ji)\pi_{ji}))T\eta \circ \epsilon^{-1}c \\
&= T((j_n^{k+1}n_{ji}\eta) \times (\widetilde{\nu}(ji)\pi_{ji}\eta))\epsilon^{-1}c \\
&= T((j_n^{k+1}l_n^k n) \times (g_k\widetilde{\nu}\pi))\epsilon^{-1}c \\
&= T((j_n^k n) \times (g_k\widetilde{\nu}\pi))\epsilon^{-1}c \\
&= T(g_k\theta_i).
\end{aligned}$$

And therefore, $\theta_{ji} = Tg_k \circ \Sigma\theta_i$. ☐

The next step is to prove that θ_i is independent from the choosing of the embedding.

Lemma 2.4.3. Let $i' : M + \partial W \hookrightarrow \mathbb{R}^{n+k}$. If k is sufficiently large (depending only on M), θ_i and $\theta_{i'}$ are homotopic.

Proof. The idea is to get a (B, f) embedding of $M \times I + W$ in $\mathbb{R}^{n+k} \times I$ agreeing with i on $M \times 0$ and with i' on $M \times 1 + \partial W$, and we use this embedding to construct the homotopy. See [St, p.20]. ☐

Note that using lemma (2.4.2) to the initial embedding i we get a k sufficiently large, and then applying lemma (2.4.3) with $W = \emptyset$, we see that $\theta_i(M)$ is independent of the choosing of i (as an element of $\lim_{k\to\infty} \pi_{n+k}(TB_k, t_0)$).

Further, suppose now that M and M' are cobordant, then there exist (B, f) manifolds W, W' with $M + \partial W \cong M' + \partial W'$. Applying lemma (2.4.3)

$$\theta(M) \sim \theta(M + \partial W) \sim \theta(M' + \partial W') \sim \theta(M').$$

And therefore, finally we have a well defined map

$$\Theta : \Omega_n(B, f) \to \lim_{k\to\infty} \pi_{n+k}(TB_k, t_0).$$

Proposition 2.4.4. Θ is a group homomorphism.

Proof. Choose $[M_1]$, $[M_2] \in \Omega_n(B, f)$ and choose k and embeddings $i_1 : M_1 \to \mathbb{R}^{n+k}$, $i_2 : M_2 \to \mathbb{R}^{n+k}$ such that M_1 and M_2 are in different half planes. Note that $\Theta(M_1 + M_2)$ is given by the composition

$$S^{n+k} \to S^{n+k} \vee S^{n+k} \xrightarrow{\Theta(M_1)\vee\Theta(M_2)} TB_k.$$

Where the first map is collapsing the equator to a point, yielding two copies of S^{n+k}. However, this composition is actually the definition of sum of the homotopy classes $\Theta(M_1) + \Theta(M_2)$. □

Proposition 2.4.5. Θ is surjective.

Proof. Choose a representative $\theta : (S^{n+k}, p) \to (TB_k, t_0)$ of a class of $\lim_{k\to\infty} \pi_{n+k}(TB_k, t_0)$. We have a map

$$Tf_k \circ \theta : (S^{n+k}, p) \to (MO(k), t_0).$$

Since $MO(k) = \lim_{s\to\infty} T\gamma^k(\mathbb{R}^{k+s})$ and $(Tf_k \circ \theta)(S^{n+k})$ is compact, exists some s such that $(Tf_k \circ \theta)(S^{n+k}) \subseteq T\gamma^k(\mathbb{R}^{k+s})$. Using (1.1.16) and the fact that $Gr_k(\mathbb{R}^{k+s})$ is a embedded submanifold of $T\gamma^k(\mathbb{R}^{k+s}) - t_0$ through the zero section. (Recall that $T\gamma^k(\mathbb{R}^{k+s}) - t_0$ is a manifold), we deform $Tf_k \circ \theta$ to a map h_k satisfying the following:

1. h_k is differentiable on the preimage of some neighborhood of $Gr_k(\mathbb{R}^{k+s})$.
2. h_0 is transverse regular on $Gr_k(\mathbb{R}^{k+s})$.
3. Setting $M = h_k^{-1}(Gr_k(\mathbb{R}^{k+s}))$ there is some tubular neighborhood N of M such that $h_k|_N$ is a bundle map (Actually N is isomorphic to the normal bundle of M).
4. There is a closed set V containing t_0 in its interior, for which $Tf_k \circ \theta$ agrees with h_k on $h_k^{-1}(V)$.

Since $h_k|_M$ classifies the normal bundle of M, we can deform it by homotopy to a map $h : (S^{n+k}, p) \to (MO(k), t_0)$, satisfying the above properties and such that

$$h|_M = \nu : M \to Gr_k(\mathbb{R}^{k+s}) \hookrightarrow BO(k),$$

and h is simply the usual translation of vectors in some normal tubular neighborhood of M.

$Tf_k : TB_k \to MO(k)$ is a fibration except in the point t_0, and $t_0 \notin Tf_k \circ \theta(S^{n+k} - h^{-1}(int(V)))$, by the covering homotopy theorem we find a homotopy

$$H_0 : (S^{n+k} - h^{-1}(int(V))) \times I \to TB_k,$$

such that $H_0 = \theta$ at 0 and $Tf_k \circ H(x,t) = h(x)$ for all $t \in I$. By (4), we may take H_0 to be pointwise fixed on the boundary of V. Thus H_0 can be extended to a homotopy

$$H : (S^{n+k}, p) \times I \to (TB_k, t_0),$$

by sending $h^{-1}(V)$ to the point p. Set $\theta_1 = H|_{S^{n+k}\times 1}$.

We have that $\theta_1^{-1}(B_k) = h^{-1}(BO(k)) = h^{-1}(Gr(R^{k+s})) = M$. Actually, $\theta_1|_M$ gives a lift of the normal map $h|_M$ since $Tf_k \circ \theta_1 = h$, and we chose h agree with the normal map of M. This makes a (B, f)

structure for M.

Now consider $\Theta(M)$ with this (B, f) structure, since we chose h to be just translation around M, through the definition of $\Theta(M)$, we can find N_ϵ such that $\theta_1|_{N_\epsilon} = \Theta(M)|_{N_\epsilon}$. Since $TB_k - B_k$ can be deformed to t_0 we can homotope θ_1 to $\Theta(M)$. So $\theta \sim \theta_1 \sim \Theta(M)$ and thus, Θ is surjective.

□

Proposition 2.4.6. Θ is injective.

Proof. Let M be a (B, f) manifold with $\Theta(M) = 0$. Then there is a k such that $\Theta(M) : (S^{n+k}, p) \to (TB_k, t_0)$ is homotopic to the constant map $\theta_0 : S^{n+k} \to t_0$ by a homotopy $H : S^{n+k} \times I \to TB_k$.

Choose H such that for some $\delta > 0$, $H|_{S^{n+k}} \times t = \Theta(M)$ for $t \leq \delta$. As the previous proposition, by compactness $Tf_k \circ H(S^{n+k} \times I) \subseteq T\gamma^k(\mathbb{R}^{k+s})$ for some s. As before, we deform $Tf_k \circ H$ to a map

$$K : S^{n+k} \times I \to MO(k),$$

which is smooth near $Gr_k(\mathbb{R}^{k+s})$, transverse regular on $Gr_k(\mathbb{R}^{k+s})$ and such that $K = Tf_k \circ H$ on $N_\epsilon \times [0, \delta]$ for some $d > 0$. By transversality , $W = H^{-1}(Gr_k(\mathbb{R}^{k+s}))$ is a submanifold of $\mathbb{R}^{n+k} \times I$. Since $K|_{S^{n+k} \times 1} = Tf_k \circ H|_{S^{n+k} \times 1}$ is the constant map at t_0 we see that $\partial W \subseteq \mathbb{R}^{k+s} \times 0$. Since $K = Tf_k \circ H$ on $N_\epsilon \times [0, \delta]$ we see that $\partial W = M$.

We have only to find a (B, f) structure on W compatible with that on M. Further homotope K to get $K|_M$ to be the normal map, and applying the covering homotopy theorem from $Tf_k \circ H$ to K, we obtain a homotopy form H to a map

$$\theta : S^{n+k} \times I \to TB_k,$$

such that $\theta|_{S^{n+k} \times t} = \Theta(M)$ for small t and $\theta|_{S^{n+k} \times 1} = \theta_0$. Actually, $\theta|_W$ is a lifting of the normal map $K|_W$. This gives a (B, f) structure on M which induces de correct one on $M = \partial W$. So $M \equiv 0$ in $\Omega_n(B, f)$ and thus Θ is injective □

Some computations on specifically (B, f) structures are:

Corollary 2.4.7.

1. $\Omega^{un} \cong \mathbb{Z}_2[x_i]$ where x_i is a generator of degree not of the form $2^k - 1$.
2. $\Omega^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[y_n]$ where y_n is a generator of degree $4n$.
3. $\Omega_n^{fr} \cong \lim_{k \to \infty} \pi_{n+k}(S^k)$. □

2.5 Determination of $\Omega^{SO} \otimes \mathbb{Q}$

Now we will determine the structure of $\Omega^{SO} \otimes \mathbb{Q}$, recall that tensoring with $\mathbb{Q}$ kills the torsion of the ring Ω^{SO}, and preserves the free structure. By the Thom–Pontryagin theorem, we have

$$\Omega_n^{SO} \cong \lim_{k \to \infty} \pi_{n+k}(TBSO(k), t_0).$$

We will use the *Rational Hurewicz theorem* [Kr].

Theorem 2.5.1. Let X be a simply connected topological space with $\pi_i(X) \otimes \mathbb{Q} = 0$ for $i \leq k$. Then the Hurewicz map

$$h \otimes \mathbb{Q} : \pi_x(X) \otimes \mathbb{Q} \to H_i(X, \mathbb{Q}),$$

induces an isomorphism for $1 \leq i \leq 2k$ and a surjection for $i = 2k + 1$.

Theorem 2.5.2. Ω_n^{SO} is finite for n not divisible by 4, and has rank $p(n/4)$ for n divisible by 4.

Proof. By the Thom–Pontryagin theorem

$$\Omega_n^{SO} \cong \lim_{k\to\infty} \pi_{n+k}(TBSO(k), t_0).$$

Choose $k > n$, by taking the limit of the finite complexes $T\widetilde{\gamma}^k(\mathbb{R}^{n+k})$ and using the previous theorem, we have that

$$\text{rank } \pi_{n+k}(TBSO(k), t_0) = \text{rank } H_{n+k}(TBSO(k), \mathbb{Z}).$$

But this is the same as the rank of $H_{n+k}(TBSO(k), t_0, \mathbb{Z})$ by the exact sequence of the pair $(TBSO(k), t_0)$. Since the $Hom(\text{-}, \mathbb{Z})$ functor preserves the free part of a group, this rank is the same as the rank of $H^{n+k}(TBSO(k), t_0, \mathbb{Z})$. By (1.1.14)

$$H^{n+k}(TBSO(k), t_0, \mathbb{Z}) \cong H^n(BSO(k), \mathbb{Z}).$$

By Corollary (1.1.26), this last is finite for n not divisible by 4 and has rank $p(n/4)$ for n divisible by 4. □

Actually, $\Omega^{SO} = \bigoplus_{n=0}^{\infty} \Omega_n^{SO}$ has a structure of a graded $\mathbb{Z}-$algebra.

Proposition 2.5.3. Ω^{SO} is a commutative graded $\mathbb{Z}$-algebra with product induced by the Cartesian product of manifolds.

Proof. If M, M', N are closed and $M \equiv M'$, there is a compact manifold W such that $\partial W = M + (-M')$. Then

$$\partial(M \times N) \cong (\partial W \times N) + (-(W \times \partial N)) \cong ((M + (-M')) \times N) + (W \times \emptyset) \cong (M \times N) + ((-M') \times N)$$

and so $M \times N \equiv M' \times N$. Analogously, if $N \equiv N'$ then $M \times N \equiv M \times N'$. Thus the Cartesian product induces a well defined product on Ω^{SO}. By the inner properties of the Cartesian product, this induced product is also commutative, associative and distributive with respect to $+$. Recall that the multiplication is graded and the identity element is the class of the manifold consisting of single point $\{\cdot\}$. □

Theorem 2.5.4. $\Omega^{SO} \otimes \mathbb{Q}$ is the free $\mathbb{Q}$-algebra generated by $\mathbb{C}P^{2n}$ for $n \geq 1$.

Proof. By (1.1.29)

$$P_{(n)}((T\mathbb{C}P^{2n})_{\mathbb{R}}) = 2n + 1$$

Let $m = 4n$ and $I = (i_1, \dots, i_k)$ be a partition of n. Define

$$M_I = \mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_k}$$

Let I' another partition of n, and by (1.1.28)

$$S_{I'}[p(M_I)] = \sum_{I_1 \cdots I_k} S_{I_1}[p(M_{I_1})] \cdots S_{I_k}[p(M_{I_k})]$$

If I' does not refine [1] I, $S_{I'}[p(M_I)]$ must be zero since there is no way to choose the partitions $I_1, \ldots, I_k$. Also, if $I' = I$, $S_I(p(M_I)) = 1$.

Consider the matrix indexed by the partitions of n ordered by the order $I \leq I'$ if I' refines I. So, these calculations show that this matrix is triangular with $1'$s on the diagonal. Therefore it has non-zero determinant and the Pontryagin numbers of the manifolds M_I are linearly independent over $\mathbb{Q}$. Since the polynomials s_I are a basis for the symmetric functions of degree n, the manifolds M_I are linearly independent over $\mathbb{Q}$ as elements of Ω_m^{SO}, and there are exactly $p(n)$ of them, so they form a basis for Ω_m^{SO}.

We conclude that the set of classes of $\mathbb{C}P^{2n}$ is algebraically independent as elements of $\Omega^{SO} \otimes \mathbb{Q}$. Since $(\Omega^{SO} \otimes \mathbb{Q})$ has no torsion, by theorem (2.5.2) it is 0 if i is not divisible by 4, and has rank $p(i/4)$ for i divisible by 4.

These ranks are the same as $(\mathbb{Q}[\mathbb{C}P^{2n}])^i$ for all i. So,

$$\Omega^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^{2n}].$$

□

Corollary 2.5.5. Let M be a compact oriented $4n$-manifold. If M is the boundary of an oriented $(4n+1)$ manifold W, then all of the Pontryagin numbers of M are zero.

Proof. There are exact sequences

$$H_{4n+1}(W, M, \mathbb{Z}) \xrightarrow{\partial} H_{4n}(M, \mathbb{Z}) \xrightarrow{i_*} H_{4n}(W, \mathbb{Z})$$

and

$$H^{4n}(W, \mathbb{Z}) \xrightarrow{i^*} H^{4n}(M, \mathbb{Z}) \xrightarrow{\delta} H^{4n+1}(W, M, \mathbb{Z}).$$

Let $\mu_{W,M} \in H_{4n+1}(W, M, \mathbb{Z})$ be the fundamental class of the pair (W, M) and $\mu_M \in H_{4n}(M, \mathbb{Z})$ the fundamental homology class of M. Then $\partial \mu_{W,M} = \mu_M$. Since there is an unique outward pointing normal vector along $M \subseteq W$, so

$$TW|_M = TM \oplus \epsilon^1.$$

Thus

$$p(TW|_M) = p(TM).$$

Therefore for any partition I of $4n$,

$$\begin{aligned} P_I[M] &= P_I[TW|_M] \\ &= \langle p_I(TW|_M), \mu_M \rangle \\ &= \langle i^* p_I(TW), \mu_M \rangle \\ &= \langle i^* p_I(TW), \partial \mu_{W,M} \rangle \\ &= \langle \delta i^* p_I(TW), \mu_{W,M} \rangle \\ &= \langle 0, \mu_{W,M} \rangle. \end{aligned}$$

By exactness. □

[1] I' *refines* I if $I' = I_1 \cdots I_k$ where each I_j is a partition of i_j

2.6 The Hirzebruch Signature Theorem

This theorem is a special case of the Atiyah-Singer theorem, it is due to Hirzebruch and it is closely related to the cobordism as an application.

Recall that if we have a quadratic form represented by a matrix A over $\mathbb{Q}$, the signature of this form is the number of positive eigenvalues minus the number of negative eigenvalues.

Definition 2.6.1. Let M be a compact oriented manifold of dimension n. The *signature* of M, $\sigma(M)$, is defined as follows: if n is not divisible by 4, then $\sigma(M) = 0$. If n is divisible by 4, say $n = 4m$, we define $\sigma(M)$ to be the signature of the rational quadratic of Q on $H^{2m}(M, \mathbb{Q})$ given by

$$Q(x) = \langle x \cup x, \mu_M \rangle \in \mathbb{Q},$$

where $\mu_M \in H_n(M, \mathbb{Q})$ is the fundamental rational homology class of M.

In the case $n = 4m$, the signature is computed by choosing a basis $x_1, \ldots, x_k$ of $H^{2m}(M, \mathbb{Q})$ for which the symmetric matrix $(\langle x_i \cup x_j, \mu_M \rangle)_{ij}$ is diagonal, we subtract the number of diagonal negative entries from the number of diagonal positive entries. This value is $\sigma(M)$.

Remark. By the Poincaré duality, the computation of the signature of a $4m$-dimensional manifold is equivalent to compute the signature of the quadratic form given by the $\cap$ (intersection) product in the $2m^{th}$-homology.

$$\begin{array}{ccc}
H^{2m}(M) \otimes H^{2m}(M) & \xrightarrow{\ \cup\ } & H^{4k}(M, \partial M) \\
\downarrow \cong & & \downarrow \cong \\
H_{2k}(M, \partial M) \otimes H_{2k}(M, \partial M) & \xrightarrow{\ \cap\ } & \mathbb{Z}
\end{array}$$

Proposition 2.6.2. The signature satisfies the following properties.

1. $\sigma(M + N) = \sigma(M) + \sigma(N)$.
2. $\sigma(M \times N) = \sigma(M)\sigma(N)$.
3. If $M = \partial W$ then $\sigma(M) = 0$.

Proof. 1. The interesting case is when both M and N have dimension $4m$. Since $H^{2m}(M + N, \mathbb{Q}) = H^{2m}(M, \mathbb{Q}) \oplus H^{2m}(N, \mathbb{Q})$ we have the result.

2. Let $W = M \times N$ and m, n, p be the respective dimensions of M, N, W. If p is not divisible by 4, then one of m or n is not zero module 4 and both sides of equality are zero.

   Suppose that $p = 4n$, then by the Künneth theorem,

$$H^{2k}(W, \mathbb{Q}) \cong \sum_{s=0}^{2k} H^s(M, \mathbb{Q}) \otimes H^{2k-s}(N, \mathbb{Q}).$$

This vector space decomposes into the subspaces

$$H^s(M,\mathbb{Q}) \otimes H^{2k-s}(N,\mathbb{Q}) \oplus H^{m-s}(M,\mathbb{Q}) \otimes H^{2k+s-m}(N,\mathbb{Q})$$

for $s < m/2$.

Let $\{v_i^s\}$ and $\{w_j^t\}$ basis for $H^s(M,\mathbb{Q})$ and $H^t(N,\mathbb{Q})$ respectively, such that $\langle v_i^s \cup v_j^{m-s}, \mu_M \rangle = \delta_{ij}$ for $s \neq m/2$ and $\langle w_i^t \cup w_j^{n-s}, \mu_N \rangle = \delta_{ij}$ for $t \neq n/2$.

Consider the group $A = H^{\frac{m}{2}}(M,\mathbb{Q}) \otimes H^{\frac{n}{2}}(N,\mathbb{Q})$ and $A = 0$ in the case m, n are odd. Recall that two elements $x, y \in H^{2k}(W,\mathbb{Q})$ are said to be orthogonal if $\langle x \cup y, \mu_W \rangle = 0$. Then A is orthogonal to the subgroup B of $H^{2k}(W,\mathbb{Q})$ which consists of all elements of the summation given by the Künneth theorem in which no elements of A occur. As a basis for the group B we can take $\{v_i^s \otimes w_j^{2k-s}\}$, $0 \leq s \leq m, s \neq n/2$. Now

$$\langle v_i^s \otimes w_j^{2k-s})(v_{i'}^{s'} \otimes w_{j'}^{2k-s'}), \mu_W \rangle = \pm 1$$

if $s + s' = m, i = i', j = j'$. And it is equal to 0 otherwise.

So with respect to this basis, the restriction of the bilinear form of W to B is represented by a matrix with block $\pm\binom{0\ 1}{1\ 0}$ on the diagonal and zero elsewhere. Therefore the signature of the restriction to B is 0. Since A and B are orthogonal, $\sigma(Q)$ is equal to signature of the restriction of the bilinear form to A.

Therefore $\sigma(W) = \sigma(M)\sigma(N)$ regardless if n and m are divisible by 4 or not.

3. The interesting case is when M is the boundary of a oriented $4m+1$-manifold W. Let $j : M \hookrightarrow W$. Consider the diagram of homomorphism

$$\begin{array}{ccccc}
H^{2k}(W,\mathbb{Q}) & \xrightarrow{j^*} & H^{2k}(M,\mathbb{Q}) & \longrightarrow & H^{2k+1}(W,M,\mathbb{Q}) \\
\downarrow & & \downarrow i & & \downarrow \\
H_{2k+1}(W,V,\mathbb{Q}) & \longrightarrow & H_{2k}(M,\mathbb{Q}) & \xrightarrow{j_*} & H_{2k}(W,\mathbb{Q})
\end{array}$$

The rows parts are exact homology and cohomology sequences and the vertical arrows are isomorphism given by the Poincaré duality. Let A^{2k} be the image of j^* in $H^{2k}(M,\mathbb{Q})$ and let K_{2k} be the kernel of j_* in $H_{2k}(M,\mathbb{Q})$. Then A^{2k} is the dual space of the quotient $H_{2k}(M,\mathbb{Q})/K_{2k}$, under the duality between $H^{2k}(M,\mathbb{Q})$ and $H_{2k}(M,\mathbb{Q})$.

Observe that for $x \in H^{2k}(M,\mathbb{Q})$,

$$x \in A^{2k} \Leftrightarrow i(x) \in K_{2k}.$$

If $b_{2k} = \dim H_{2k}(M,\mathbb{Q})$ is the $2k-th$ betti number of V,

$$\dim A^{2k} = \dim K_{2k} = b_{2k} - \dim K_{2k}$$

and

$$\dim A^{2k} = \frac{1}{2} b_{2k}.$$

If $x = j^* y \in A^{2k}$, then

$$\langle x^2, \mu_M \rangle = \langle j^*(y^2), \mu_M \rangle = \langle y^2, j_* \mu_M \rangle = 0.$$

Therefore the set $\{x \in H^{2k}(M, \mathbb{Q}) : \langle x^2, \mu_M \rangle = 0\}$ contains the subspace A^{2k} of dimension $\frac{1}{2} b_{2k}$. It follows that the bilinear from over M has equal number of positive and negative eigenvalues and hence $\sigma(M) = 0$.

□

As an important result from these properties is the fact that if M and N are manifolds such that $M \equiv N$ in the oriented cobordism, then $\sigma(M) = \sigma(N)$. So σ induces a well defined $\mathbb{Q}$-algebras homomorphism

$$\sigma : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}.$$

Now we construct another homomorphism. Let $A = \mathbb{Q}[t_1, t_2, \ldots]$ be a graded commutative $\mathbb{Q}$-algebra where t_i has degree i. Define an associated ring $\mathcal{A}$ to be the ring of infinite formal sums

$$a = a_0 + a_1 + \cdots$$

where $a_i \in A$ is homogeneous of degree i. Let $\mathcal{A}^+$ be the subgroup of $\mathcal{A}$ of elements with leading term 1.

Definition 2.6.3. Let $K_1(t_1), K_2(t_1, t_2), K_3(t_1, t_2, t_3) \cdots \in A$ a sequence of polynomials where K_n is homogeneous of degree n. For $a = 1 + a_1 + \cdots \in \mathcal{A}^+$ we define $K(a) \in \mathcal{A}^+$ by

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots$$

We say that K_n form a *multiplicative sequence* if $K(ab) = K(a)K(b)$ for all $a, b \in \mathcal{A}^+$.

Example 2.6.4. A simple example is provided by the sequence

$$K_n(t_1, \ldots, t_n) = \lambda^n t_n$$

for any $\lambda \in \mathbb{Q}$.

A more interesting example is the following. Consider the power series expansion of the function

$$\frac{\sqrt{t}}{\tanh \sqrt{t}} = 1 + \frac{1}{3} t - \frac{1}{45} t^2 + \cdots + (-1)^{i-1} \frac{2^{2i} B_i}{(2i)!} t_i + \cdots$$

where B_i is the i^{th} Bernoulli number. Set

$$\lambda_i = (-1)^{i-1} \frac{2^{2i} B_i}{(2i)!}.$$

For any partition $I = (i_1, \ldots, i_k)$ of n, set $\lambda_I = \lambda_{i_1} \cdots \lambda i_k$. Now define polynomials $L_n(t_1, \cdots, t_n) \in A$ by

$$L_n(t_1, \ldots, t_n) = \sum_I \lambda_I s_I(t_1, \ldots, t_n)$$

where the sum is over all partitions of n and s_I is the polynomial of (1.1.27).

Lemma 2.6.5. The set of polynomial L_n form a multiplicative sequence.

Proof. From the definition of s_I we have that L_n is homogeneous of degree n. Let $a, b \in \mathcal{A}^+$, then

$$\begin{aligned} L(ab) &= \sum_I \lambda_I s_I(ab) \\ &= \sum_I \sum_{I_1 I_2 = I} s_{I_1}(a) s_{I_2}(b) \\ &= \sum_{I_1 I_2 = I} \lambda_{I_1} \lambda_{I_2} s_{I_1}(a) s_{I_2}(b) \\ &= L(a)L(b). \end{aligned}$$

□

Note that the coefficient of t_1^n in L_n is λ_n, since the only $s-$polynomial containing that monomial is $s_{(n)}$.

Definition 2.6.6. Let M be a manifold of dimension n. We define the L-genus of M, $L(M)$ as follows. If n is not divisible by 4, then $L[M] = 0$. If $n = 4m$, then we define

$$L(M) = \langle L_m(p_1(M), \ldots, p_m(M)), \mu_M \rangle.$$

Proposition 2.6.7. The assignation $[M] \mapsto L(M)$ defines a $\mathbb{Q}-$algebra homomorphism

$$L : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}.$$

Proof. The additivity of the correspondence is immediate. From corollary (2.5.5), follows that the L-genus of a boundary is zero. This two facts together guarantee that L is well defined.

Consider a product manifold $W = M \times N$, since the total Pontryagin class $p(M \times N) = p(M) \times p(N)$, up to elements of order 2, we have that

$$L(p(W)) = L(p(M)) \times L(p(N)).$$

Therefore

$$\begin{aligned} L(W) &= \langle L(p(W)), \mu_W \rangle \\ &= \langle L(p(M)) \times L(p(N)), \mu_M \times \mu_N \rangle \\ &= \langle L(p(M)), \mu_M \rangle \langle L(p(N)), \mu_N \rangle \\ &= L(M)L(N). \end{aligned}$$

□

The Hirzebruch signature theorem states that the two homomorphism constructed in this section coincide.

Theorem 2.6.8 (Hirzebruch Signature Theorem)**.** Let M be an oriented manifold. Then $\sigma(M) = L(M)$.

Proof. Since both σ and L are $\mathbb{Q}$-algebra homomorphism from $\Omega^{SO} \otimes \mathbb{Q}$ to $\mathbb{Q}$, it will suffice to show that it is true on the set of generators of $\Omega^{SO} \otimes \mathbb{Q}$, which by theorem (2.5.4) they are the complex projective spaces $\mathbb{C}P^{2n}$.

Recall that $H^{2n}(\mathbb{C}P^{2n}, \mathbb{Z})$ is generated by a element x^n, with $x \in H^2(\mathbb{C}P^{2n}, \mathbb{Z})$, then

$$\langle x^n \cup x^n, \mu_{\mathbb{C}P^{2n}} \rangle = (-1)^{2k} = 1.$$

Thus $\sigma(\mathbb{C}P^{2n}) = 1$.

Now, from [MS, p.177] we have that $p((T\mathbb{C}P^{2n})_{\mathbb{R}}) = (1 + x^2)^{2n+1}$. Since the coefficient of t_1^i in L_i is λ_i, then

$$\begin{aligned} L(1 + x^2) &= 1 + L(x^2) + L_2(x^2, 0) + L_3(x^2, 0, 0) + \cdots \\ &= 1 + \lambda_1 x^2 + \lambda_2 x^4 + \cdots \\ &= \frac{\sqrt{x^2}}{\tanh \sqrt{x^2}} \\ &= \frac{x}{\tanh x}. \end{aligned}$$

Therefore,

$$\begin{aligned} L(p(T\mathbb{C}P^{2n})_{\mathbb{R}}) &= L((1 + x^2)^{2n+1}) \\ &= L(1 + x^2)^{2n+1} \\ &= \Big(\frac{x}{\tanh x}\Big)^{2n+1}. \end{aligned}$$

Thus, $L_n(p(T\mathbb{C}P^{2n})_{\mathbb{R}})$ is equal the x^{2n} term in this expansion, and $L(\mathbb{C}P^{2n})$ is simply the coefficient of that term.

We compute that coefficient by methods of complex analysis. The coefficient of z^{2n} in the Taylor expansion of $(z/\tanh z)^{2n+1}$ by the Cauchy Integral Formula is

$$\frac{1}{2\pi i} \oint \Big(\frac{z}{\tanh z}\Big)^{2n+1} \frac{dz}{z^{2n+1}} = \frac{1}{2\pi i} \oint \frac{dz}{(\tanh z)^{2n+1}}.$$

Make the substitution $u = \tanh z$, so

$$dz = \frac{du}{1 - u^2} = (1 + u^2 + u^4 + \cdots)du.$$

Thus,

$$\begin{aligned} \frac{1}{2\pi i} \oint \frac{dz}{(\tanh z)^{2n+1}} &= \frac{1}{2\pi i} \oint \frac{(1 + u^2 + u^4 + \cdots)du}{u^{2n+1}} \\ &= \frac{1}{2\pi i} \oint \frac{du}{u} \\ &= 1. \end{aligned}$$

So $L(\mathbb{C}P^{2n}) = 1 = \sigma(\mathbb{C}P^{2n})$.

$\square$

Example 2.6.9. In particular, the coefficient s_m of $p_m(M)$ in L_m is given by

$$s_0 = 1 \quad \text{and} \quad s_m = \frac{2^{2m}(2^{2m-1}-1)B_m}{(2m)!}.$$

For example

$$\begin{aligned}
\sigma(M^4) &= \langle \frac{p_1(M)}{3}, \mu_M \rangle \\
\sigma(M^8) &= \langle \frac{7p_2(M) - p_1^2(M)}{45}, \mu_M \rangle \\
\sigma(M^{12}) &= \langle \frac{62p_3(M)}{945} + \cdots, \mu_M \rangle \\
\sigma(M^{16}) &= \langle \frac{127p_4(M)}{4725} + \cdots, \mu_M \rangle,
\end{aligned}$$

where the dots indicate a rational function in $p_1(M), \ldots, p_{m-1}(M)$.

Chapter 3

Construction of Exotic Spheres

In this chapter we will construct manifolds which are homeomorphic to the sphere, but we will also show that these manifolds are not diffeomorphism to the sphere. This kind of Manifolds are known as *exotic spheres* and its existence showed that the "smooth Poincaré conjecture" does not hold. Mainly, we will construct manifolds with an certain invariant non-zero (Signature, Arf-Kervaire) and with its boundary homeomorphic (or just homotopic by Poincaré's Conjecture) to a sphere. In suitable dimensions this fact will guarantee that these boundaries are exotic spheres.

3.1 S^3-bundles over S^4

The idea is to construct spherical bundles $S^{n-1} \to M \to S^n$, and by the long exact sequence in homotopy associated to this fibration, we get that M is a $(2n-2)$-connected manifold, and one can guarantee that this manifold is indeed homeomorphic to the sphere S^{2n-1}. This was the first example of an exotic structure over the spheres and it is due to J. Milnor [M1].

We start with some examples where the manifold M is actually the S^{2n-1} sphere as we know it.

Example 3.1.1. 1. The trivial case is to consider the fibration $S^0 \to S^1 \xrightarrow{\pi} S^1$ given by $\pi(z) = z^2$.

2. A more interesting example is the following. Describe S^3 as $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and consider the map

$$\begin{aligned} h_1 : S^3 &\to \mathbb{C} \cup \infty \\ (z_1, z_2) &\mapsto \overline{z_1 z_2^{-1}} \end{aligned}$$

and compose it with the inverse of the stereographic map to obtain a map $\pi : S^3 \to S^2$. It is not difficult to see that $\pi(z_1, z_2) = \pi(e^{i\theta} z_1, e^{i\theta} z_2)$ and thus we get a fibration

$$S^1 \to S^3 \to S^2$$

3. Recall that $\mathbb{H}$ denotes the quaternions and its elements are described by the form $q = a_1 + a_2 i + a_3 j + a_4 k$ with $a_i \in \mathbb{R}$. Also can be described by $q = c_1 + c_2 j$ where $c_1, c_2 \in \mathbb{C}$.

Consider $S^7 = \{(a, b, c, d) \in \mathbb{C}^4 : |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1\}$, and let $q_1 = a + bj$ and $q_2 = c + dj$. As in the previous example, consider the map

$$h : S^7 \to \mathbb{C}^2 \cup \{\infty\}$$

given by $h_1(q_1, q_2) = \overline{q_1 q_2^{-1}}$ and compose it with the inverse stereographic projection to get a fibration

$$S^3 \to S^7 \to S^4.$$

If the base space of any fibration will be the sphere S^n, there is an easy way to construct many different fiber bundles, where the typical fiber will be a topological space F. Let G denote the group of homeomorphism of F and let $f : S^{n-1} \to G$.

We can construct the total space

$$E_f = (U_0 \times F) \sqcup (U_1 \times F)/\sim$$

where $(u, y) \sim (u, f \circ \pi(u)y)$ is an equivalence relation and $U_0 = S^n - \{\text{north pole}\}$, $U_1 = S^n - \{\text{south pole}\}$ and $\pi : U_0 \cap U_1 \to S^{n-1}$ is the projection onto the equator. Observe that if f is smooth, we can equip E_f with a smooth structure.
The following theorem identifies all equivalence classes of fiber bundles over a sphere.

Theorem 3.1.2. For a fixed fiber F, all F-bundles over S^n are isomorphic to one obtained by the previous construction, and two such bundles are isomorphic if and only if the defining maps $S^{n-1} \to G$ are homotopic, where G denotes the group of homeomorphism of F. □

As an immediate corollary, we get that the F-bundles over S^n are classified by the group $\pi_{n-1}(G)$.

Now we will focus on the case $B = S^4$, $F = S^3$ and $G = SO(4)$. So we can classify such bundles by $\pi_3(SO(4))$.

Proposition 3.1.3. There is an isomorphism between the groups $\pi_3(SO(4))$ and $\mathbb{Z} \oplus \mathbb{Z}$.

Proof. We consider S^3 as the unit sphere in the space of quaternions $\mathbb{H}$. Define the map

$$SO(4) \to S^3 \times SO(3)$$

given by $\phi \mapsto (\phi(1), \phi(1)^{-1}\phi)$, it is well defined since ϕ preserves the norm. Here we set $SO(3)$ as the subgroup of $SO(4)$ which fixes 1, and so $\phi(1)^{-1}\phi \in SO(3)$. We can construct an inverse

$$S^3 \times SO(3) \to SO(4)$$

by setting $(u, \psi) \mapsto \phi_{u,\psi}$ and $\phi_{u,\psi}(v) = u\psi(v)$.

So, there is a homeomorphism between $SO(4)$ and $S^3 \times SO(3)$. Recall that $SO(3) \cong \mathbb{R}P^3$ and there exist a 2-sheeted covering $S^3 \xrightarrow{\rho} \mathbb{R}P^3$, getting a fibration

$$\mathbb{Z}_2 \to S^3 \to \mathbb{R}P^3,$$

thus $\mathbb{Z} \cong \pi_3(S^3) \cong \pi_3(\mathbb{R}P^3)$. Combining this result with the last homeomorphism we get then

$$\pi_3(SO(4)) \cong \pi_3(S^3) \oplus \pi_3(SO(3)) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

□

We can describe explicitly all the equivalence classes of S^3-bundles over S^4 (with respect to the structural group $SO(4)$).

The elements of $\pi_3(S^3)$ can be represented by maps $\phi_a : S^3 \to S^3$, $a \in \mathbb{Z}$, which are of the form $\phi_a(z) = z^a$.

Since ρ is a finite cover, and actually an isomorphism on π_3, the elements of $\pi_3(SO(3))$ can be represented by $\rho \circ \phi_b : S^3 \to SO(3)$, $b \in \mathbb{Z}$.

Therefore, for every $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$, we have a map

$$\begin{array}{lll} S^3 & \to S^3 \times SO(3) & \to SO(4) \\ z & \mapsto (z^a, \rho(z^b)) & \mapsto (v \mapsto z^{a+b} v z^{-b}). \end{array}$$

So we have proved

Proposition 3.1.4. The map

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} &\to \pi_3(SO(4)) \\ (h, j) &\mapsto \phi_{hj} \end{aligned}$$

where $\phi_{hj}(z)(v) = z^h v z^j$ is a group isomorphism.

Denote by ξ_{hj} the S^3-bundle associated to the map ϕ_{hj} and let E_{hj} the total space of this bundle. Also, we can consider a D^4-bundle associated to this map, as before, we consider the total space

$$B_{hj} = (D^4 \times S^)) \cup (D^4 \times S^)/ \sim$$

where $(t, x) \sim (t, \phi_{hj}(t)(x))$ if $||t|| = 1$. And so, the boundary of this bundle is a S^3-bundle and $\partial B_{hj} = E_{hj}$

Example 3.1.5. 1. $B_{00} \cong D^4 \times S^4$ and $E_{00} \cong S^3 \times S^4$ since in the equator of S^4, $\phi_{00} = id$.

2. $B_{10} \cong \mathbb{H}P -$ open disc and $E_{10} \cong S^7$.

Recall that $\mathbb{H}P^n$ is the quotient of $\mathbb{H}^{n+1} - \{0\}$ under the identification $(u, v, w) \sim (xu, xb, xw)$ for $x \in \mathbb{H}^*$. There is a natural injection of $\mathbb{H}P^1 \hookrightarrow \mathbb{H}P^2$ and there is also a natural fibration

$$\mathbb{H} \to \mathbb{H}P^2 - \{[0, 0, 1]\} \xrightarrow{\pi} \mathbb{H}P^1$$

given by $\pi^{-1}([u, v]) = \{[u, v, w] : w \in \mathbb{H}\}$.

Notice that $\mathbb{H}P^1$ may be decomposed into two 4-disc

$$D_1 = \{[u, 1] : ||u|| \leq 1\}$$

$$D_2 = \{[1, v] : ||v|| \leq 1\},$$

sewn together through the reflection map $[u, 1] \mapsto [1, u^{-1}]$. So the fibration is an $\mathbb{H}$ (or $\mathbb{R}^4$)-bundle over S^4.

The local trivialization over D_1 and D_2 are given by

$$\phi_1 : D_1 \times \mathbb{H} \to \pi^{-1}(D_1), \phi([u, 1], w) = [u, 1, w]$$

$$\phi_2 : D_2 \times \mathbb{H} \to \pi^{-1}(D_2), \phi([1, v], w) = [1, v, w].$$

The transition map acts over the set where $[u, 1] = [1, v]$, that is, the equatorial S^3 and we have

$$\phi_2^{-1}\phi_1([u, 1], w) = \phi_2^{-1}([u, 1, w]) = \phi_2^{-1}([1, u^{-1}, u^{-1}w]) = ([1, v], vw)$$

and so $\phi_2^{-1}\phi_1$ is equal to ϕ_{10}.

From the total space $\mathbb{H}P^2 - \{[0, 0, 1]\}$ we remove the open 8-disc $\{[u, v, 1] : ||u||^2 + ||v||^2 < 1\}$ centered at $[0, 0, 1]$. So we restrict the fiber over $[u, v]$ to the set $\{[u, v, w] : ||w||^2 \leq ||u||^2 + ||v||^2\}$ and so this fiber is homeomorphic to D^4 for $[u, v]$ fixed.

Therefore, $B_{10} \cong \mathbb{H}P^2 \setminus$ open 8-disc. Moreover,$E_{10} \cong \partial B_{10} \cong S^7$ the boundary of the removed disc

We are almost done in the construction of exotic spheres, we use the total spaces of these S^3-bundles for special choosing of h, j and the Morse theory and characteristic classes will guarantee that these manifolds will be homeomorphic but not diffeomorphic to the sphere S^7 respectively.

Proposition 3.1.6. There is a group homomorphism between $\pi_{m-1}(SO(n))$ and $\pi_m(Gr_n(\mathbb{R}^{2n}))$.

Proof. Given $f, g : S^{m-1} \to SO(n)$, a representative of the sum in $\pi_{m-1}(SO(n))$ is given by $(f \vee g) \circ p$ where $p : S^{m-1} \to S^{m-1} \vee S^{m-1}$ is the pinching map.

Let Ff denotes the map $S^m \to Gr_n(\mathbb{R}^{2n})$ which classifies the n-plane bundle over S^m, induced by f.

We have the diagram

$$\begin{array}{ccccc} S^{m-1} & \xrightarrow{p} & S^{m-1} \vee S^{m-1} & \xrightarrow{f \vee g} & SO(n) \\ \downarrow & & \downarrow & & \\ S^m & \xrightarrow{p} & S^m \vee S^m & \xrightarrow{Ff \vee Fg} & Gr_n(\mathbb{R}^{2n}) \end{array}$$

The lower row represents the sum $[Ff] + [Fg]$ in $\pi_m(Gr_n(\mathbb{R}^{2n}))$. Since $F(f \vee g) = Ff \vee Fg$, it also represents $F([f] + [g])$.

□

Now we can compute the Euler and Pontryagin classes of the bundles E_{hj}

Theorem 3.1.7. The Euler and Pontryagin classes of E_{hj} are given by

$$e(E_{hj}) = \pm(h+j)\iota$$
$$p_1(E_{hj}) = \pm 2(h-j)\iota$$

where $\iota \in H^4(S^4, \mathbb{Z})$ is the generator.

Proof. Notice that $p_1(E_{hj})$ and $e(E_{hj})$ are linear in h and j. This is because the map which assigns to (h, j) the class $p_1(E_{hj}), e(E_{hj}) \in H^4(S^4, \mathbb{Z})$ are the composition of the group homomorphisms.

$$\mathbb{Z}^2 \xrightarrow{\cong} \pi_3(SO(4)) \to \pi_4(Gr_4(\mathbb{R}^8)) \to H^4(S^4, \mathbb{Z})$$

where the last map are given by $[f] \mapsto p_1(f^*(\gamma^4(\mathbb{R}^8)))$ and $[f] \mapsto e(f^*(\gamma^4(\mathbb{R}^8)))$ according to the case.

Consider the effect of reversing the fiber orientation. Interpreting S^3 as the unit quaternions, this is equivalent to conjugation by the map $v \overset{g}{\mapsto} v^{-1}$, then

$$g^{-1}(\phi_{hj})g(v) = (u^h v^{-1} u^j)^{-1} = u^{-j} v u^{-h}$$

and so E_{hj} becomes E_{-j-h}. But reversing orientation is detected by e and not by p_1, and thus $e(E_{hj}) = -e(E_{-j-h})$ and $p_1(E_{hj}) = p_1(E_{-j-h})$.

So using linearity we have that

$$e(E_{hj}) = k_1(h+j)\iota$$
$$p_1(E_{hj}) = k_2(h-j)\iota,$$

for some yet undetermined constants.

Apply the Gysin sequence to the bundle E_{10}, yielding an exact sequence

$$H^3(E_{10}, \mathbb{Z}) \to H^0(S^4, \mathbb{Z}) \xrightarrow{\cup e(E_{10})} H^4(S^4, \mathbb{Z}) \to H^4(E_{10}, \mathbb{Z}).$$

We already know that $E_{10} \cong S^7$ and so the first and last groups above are zero. Thus $e(E_{10})$ must be a generator and so $k_1 = \pm 1$.

Now we calculate k_2. Here we use the computations of $p_1(S^4) = 0$ and $p_1(\mathbb{H}P^2) = 2\beta$ (see [MS]). Recall that $B_{10} = \mathbb{H}P^2 - D^8$, and so the map $i : B_{10} \hookrightarrow \mathbb{H}P^2$ induces an isomorphism $i^* : H^4(\mathbb{H}P^2, \mathbb{Z}) \to H^4(B_{10}, \mathbb{Z})$ by the exact sequence and excision applied to the pair $(\mathbb{H}P^2, B_{10})$. Similarly, the projection $B_{10} \xrightarrow{\pi} S^4$ induces an isomorphism in the cohomology H^4.

Let α denote the generator of $H^4(B_{10}, \mathbb{Z})$ and β the generator of $H^4(\mathbb{H}P^2, \mathbb{Z})$. Consider the tangent bundle TB_{10} which is naturally isomorphic to the Whitney sum of the sub-bundle of those vectors parallel to the fiber, and those parallel to the $0-$section. That is, we have an isomorphism

$$TB_{10} \cong \pi^* E_{10} \oplus \pi^* TS^4.$$

So,

$$\begin{aligned} p_1(TB_{10}) &= \pi^*(p_1(E_{10} \oplus TS^4)) \\ &= \pi^*(p_1(E_{10}) + p_1(TS^4)) \\ &= \pi^*(p_1(E_{10})) + 0. \end{aligned}$$

Since π^* and i^* are isomorphism in H^4

$$\begin{aligned} p_1(E_{10}) &= (\pi^*)^{-1}(p_1(TB_{10})) \\ &= (\pi^*)^{-1}(i^*(p_1(\mathbb{H}P^2))) \\ &= (\pi^*)^{-1}(i^*(2\beta)) \\ &= (\pi^*)^{-1}(\pm 2\alpha) \\ &= \pm 2\iota. \end{aligned}$$

And thus $k_2 = \pm 2$. □

Now we restrict to a special choosing of h and j. Set M_k the total space E_{hj} with $h+j=1$ and $h-j=k$, the first condition will imply that M_k is homeomorphic to S^7 and the second one will show that M_k is not diffeomorphic to this sphere.

Theorem 3.1.8. M_k is homeomorphic to S^7.

Proof. Since by construction M_k is a 7-dimensional compact manifold, we show that this manifold is homeomorphic to S^7 constructing a Morse function over M_k with exactly two critical points.[1]
Recall that M_k has two charts $U_1 = (S^4 - \{N\} \times S^3) \cong \mathbb{R}^4 \times S^3$ and $U_2 = (S^4 - \{S\} \times S^3) \cong \mathbb{R}^4 \times S^3$. The transition function over $U_1 \cap U_2 \cong \mathbb{R}^4 - \{0\} \times S^3$ are then

$$\begin{aligned} U_1 \cap U_2 &\to U_1 \cap U_2 \\ (u,v) &\mapsto \Big(\frac{u}{||u||^2}, \phi_{hj}\Big(\frac{u}{||u||}\Big)(v)\Big). \end{aligned}$$

Consider the map defined over the first chart

$$g(u,v) = \frac{Re(v)}{\sqrt{1+||u||^2}} \qquad (u,v) \in U_1.$$

And in the second chart define

$$g(u',v') = \frac{Re(u'v'^{-1})}{\sqrt{1+||u'v'^{-1}||}} \qquad (u',v') \in U_2.$$

Actually, g is well defined over the whole M_k, only we have to prove that $g(u,v) = g(u',v')$ under the change of coordinates $(u,v) \mapsto (u',v)$ given by the transition map defined in $U_1 \cap U_2$.

[1]**Theorem**: Let M be a n-dimensional compact manifold. If there exist a Morse function $f : M \to \mathbb{R}$ with only two critical points, then there exists a homeomorphism of M onto S^n which is a diffeomorphism except possibly at one point.

Since $h+j=1$ and for $q \in \mathbb{Q}$ $Re(q) = Re(q^{-1})$, we have

$$\begin{aligned} Re(u'v'^{-1}) &= Re\Big(\frac{u}{||u||^2}\phi_{hj}\big(\frac{u}{||u||}(v)\big)\Big) \\ &= Re\Big(\frac{u}{||u||^2}[(\frac{u}{||u||})^h v(\frac{u}{||u||})^j]^{-1}\Big) \\ &= Re\Big(\frac{1}{||u||}(\frac{u}{||u||})^{1-(h+j)}v^{-1}\Big) \\ &= Re\Big(\frac{v^{-1}}{||u||}\Big) \\ &= \frac{1}{||u||}Re(v^{-1}) \\ &= \frac{1}{||u||}Re(v) \end{aligned}$$

On the other hand

$$1+||u'v'^{-1}||^2 = 1+||\frac{u}{||u||^2}(\frac{u}{||u||})^{-j}v^{-1}(\frac{u}{||u||})^{-h}||^2 = 1+\frac{1}{||u||^2}.$$

And this together with the previous equality,

$$g(u',v') = \frac{Re(u'v'^{-1})}{\sqrt{1+||u'v'^{-1}||^2}} = \frac{||u||Re(u'v'^{-1})}{\sqrt{1+||u||^2}} = \frac{Re(v)}{\sqrt{1+||u||^2}} = g(u,v).$$

Let us determine the critical points of f. Choose coordinates $u=(x^1,x^2,x^3,x^4)$ and $v=(y^1,y^2,y^3,y^4)$ with $||v||=1$, then

$$g(u,v) = \frac{(1-(y^2)^2-(y^3)^2-(y^4)^2)^{1/2}}{(1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2)^{1/2}}.$$

We find by calculation

$$\frac{\partial g}{\partial y^i} = \frac{-y^i}{(1-(y^2)^2-(y^3)^2-(y^4)^2)(1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2)^{1/2}}$$

$$\frac{\partial g}{\partial x^i} = \frac{(1-(y^2)^2-(y^3)^2-(y^4)^2)^{1/2}}{(1+(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2)^{3/2}}$$

and thus $dg=0$ if and only if $y^i=0$, $i=2,3,4$, $x^i=0$, $i=1,2,3,4$ yielding the critical points $(u,v)=(0,\pm 1)$. In the other chart can be verified that dg never vanishes and so we get no critical points there.

By explicit calculation we find that

$$\frac{\partial^2 g}{(y^i)^2}|_{(0,\pm1)} = \frac{\partial^2 g}{\partial y^j \partial y^i}|_{(0,\pm1)} = \frac{\partial^2 g}{\partial x^j \partial y^i}|_{(0,\pm1)} = \frac{\partial^2 g}{\partial x^j \partial x^i}|_{(0,\pm1)} = 0$$

$$\frac{\partial^2 g}{(x^i)^2}|_{(0,\pm1)} = 1.$$

That is, the critical points are non-degenerated and so g is a Morse function with only two critical points, thus M_k is indeed homeomorphic to the sphere S^7. □

We define an invariant under cobordism λ which will guarantee that M_k is in general non diffeomorphic to the sphere S^7.

Definition 3.1.9. Let M be a oriented 7-dimensional smooth manifold such that $H_3(M,\mathbb{Z}) = H_4(M,\mathbb{Z}) = 0$ and suppose that exists an oriented manifold B such that $\partial B = M$. Choose $\nu \in H_8(B, M, \mathbb{Z})$ and $\mu \in H_7(M,\mathbb{Z})$ orientations with $\nu|_M = \mu$. Since $i^*H^4(B, M, \mathbb{Z}) \to H^4(B,\mathbb{Z})$ is an isomorphism, consider the cohomology element $(i^*)^{-1}(p_1(B)) \in H^4(B, M)$.
Define

$$q(B) = \langle (i^*)^{-1}(p_1(B))^2, \nu \rangle$$

and

$$\lambda(M) = 2q(B) - \sigma(B) \text{ mod } 7.$$

Notice that for the case $M = S^7$, we choose $B = D^8$ and so $\lambda(M) = 0$ since $H^4(D^8) = 0$.

Proposition 3.1.10. $\lambda(M)$ does not depend of the choice of the manifold B

Proof. Let B_1, B_2 be manifolds such that $\partial B_1 = \partial B_2 = M$ and let ν_1, ν_2 be the respective orientations which induces the correct orientation over M. We glue these manifolds through M to obtain a manifold C without boundary.

Let ν orientation on C that induces the orientation ν_1 on B_1 and $-\nu_2$ on B_2. By the Hirzebruch signature theorem

$$\sigma(C) = \langle \frac{7p_2(C) - p_1(C)^2}{45}, \nu \rangle$$

and by bilinearity

$$45\sigma(C) + \langle p_1(C)^2, \nu \rangle = 7\langle p_2(C), \nu \rangle \equiv 0 (\text{mod } 7),$$

or equivalently

$$2\langle p_1(C)^2, \nu \rangle - \sigma(C) \equiv 0 (\text{mod } 7).$$

Consider the following diagram in which the isomorphism in the columns derived from the exact cohomology sequences and the isomorphism in the bottom row from the Mayer-Vietoris sequence

$$\begin{array}{ccc}
H^4(B_1, M, \mathbb{Z}) \oplus H^4(B_2, M, \mathbb{Z}) & \xleftarrow{h} & H^4(C, M, \mathbb{Z}) \\
\downarrow{\scriptstyle i_1^* \oplus i_2^*} & & \downarrow{\scriptstyle j^*} \\
H^4(B_1, \mathbb{Z}) \oplus H^4(B_2, \mathbb{Z}) & \xleftarrow{k} & H^4(C, \mathbb{Z})
\end{array}$$

So h is an isomorphism. For $\alpha \in H^4(C,\mathbb{Z})$ there exists $(\alpha_1, \alpha_2) \in H^4(B, M, \mathbb{Z}) \oplus H^4(B_2, M, \mathbb{Z})$ such that $j^*h^{-1}(\alpha_1, \alpha_2) = \alpha$. Then

$$\langle \alpha^2, \nu \rangle = \langle j^*h^{-1}(\alpha_1, \alpha_2)^2, \nu \rangle = \langle h^{-1}(\alpha_1, \alpha_2)^2, j_*\nu \rangle = \langle \alpha_1^2, \nu_1 \rangle - \langle \alpha_2^2, \nu_2 \rangle.$$

That is, the quadratic form of C is the direct sum of the form of B_1 and minus the form of B_2, thus their respective signatures satisfy

$$\sigma(C) = \sigma(B_1) - \sigma(B_2)$$

Now in this process take $\alpha = (i_1^*)^{-1}(p_1(B_1))$ and $\beta = (i_2^*)^{-1}(p_1(B_2))$. By naturality of the Pontryagin class, $k^*(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$, so

$$j^* h^{-1}(p_1(B_1) \oplus p_1(B_2)) = p_1(C).$$

Which implies that

$$\langle p_1(C)^2, \nu \rangle = \langle \alpha^2, \nu_1 \rangle - \langle \beta^2, \nu_2 \rangle$$

or equivalently

$$q(C) = q(B_1) - q(B_2).$$

Summarizing

$$(2q(B_1) - \sigma(B_1)) - (2q(B_2) - \sigma(B_2)) = 2q(C) - \sigma(C) \equiv 0 (\text{mod } 7).$$

□

Now we can use λ to distinguish between some of the M_k and S^7. To that, we have this final result.

Theorem 3.1.11. $\lambda(M_k) \equiv k^2 - 1 \bmod 7$.

Proof. Consider the bundle $D^4 \to B_k \to S^4$, we have an isomorphism $H^*(B_k, \mathbb{Z}) \cong H^*(S^4, \mathbb{Z})$, so $H^4(B_k, \mathbb{Z})$ is cyclic and $\sigma(B_k)$ must be equal to ± 1. Let ν an orientation over B_k such that $\sigma(B_k) = 1$. Exactly as in the proof of theorem (3.1.7), there is a bundle isomorphism $TB_k \cong \pi^*(M_k \oplus TS^4)$. Thus

$$p_1(B_k) = \pi^*(\pm 2k\iota + 0) = \pm 2k\pi^*\iota.$$

Therefore

$$\lambda(M_k) = 2q(B_k) - \sigma(B_k) = 2\langle \pm(2k\pi^*\iota)^2, \nu \rangle - 1 = 2\langle 4k^2\pi^*\iota^2, \nu \rangle - 1 = 8k^2 - 1 \equiv k^2 - 1 (\text{mod } 7).$$

□

Since $\lambda(M_3) = 1$, $\lambda(M_5) = 3$ and $\lambda(M_7) = 6$.

Corollary 3.1.12. The manifolds M_3, M_5 and M_7 are exotic spheres.

3.2 Plumbing of Disk Bundles

In this section, we will construct a manifold W^{4n}, with boundary a homotopy sphere, which will have the following intersection matrix:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

This matrix satisfies $\det(A) = 1$ and $\sigma(A) = 8$.

Recall that the oriented n-plane bundles over a sphere of dimension k are classified by the group $\pi_k(SO(k))$, and from each n-plane bundle we can obtain a n-disk bundle over S^k.

Definition 3.2.1. The *plumbing* of two disk bundles are made following this steps:

1. Suppose that we have two such disk bundles
$$\alpha : D^t \to E_\alpha \to S^r$$
$$\beta : D^r \to E_\beta \to S^t$$
2. Choose open sets $U_r \subseteq S^r$ and $U_t \subseteq S^t$, by trivial locality condition, there are diffeomorphism
$$\alpha|_{U_r} \cong D^r \times D^t$$
$$\beta|_{U_t} \cong D^t \times D^r$$
3. Now use this diffeomorphism to make an identification between the fibre disk of α, D^t_α, with the base disk of β, D^t_β. (And viceversa)

This manifold is said to be the result of plumbing α and β.

Choose diffeomorphism
$$\theta_1 : D^r_\alpha \to D^r_\beta \qquad\qquad \theta_2 : D^r_\alpha \to D^r_\beta.$$
The diffeomorphism θ_1 and θ_2 can be chosen to either both preserve or reverse orientation. We say we plumb with sign $+1$ if both θ_1 and θ_2 are orientation preserving, and sign -1 if both are orientation reversing. Note that the result of plumbing two disk bundles is oriented compatible with the given orientation, regardless of sign, if at least one of r or t is even.

Remark. We can represent the plumbing by a diagram in the following way. For each bundle, draw a dot, and label them with the corresponding element in $\pi_k(SO(n))$. Each time we plumb two of these bundles together, join the appropriate dots with a line. Label this line with the sign of plumbing.

Restrict now to use only stably-trivial bundles with the dimension of the base space equals to the fiber dimension(a even integer). Associate to the graph a symmetric matrix A over $\mathbb{Z}$ with even entries on the diagonal.

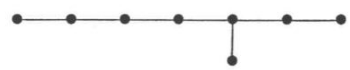
Begin with n bundles over the k-sphere and arrange these in some order. Suppose that the i^{th} bundle is represented by $\lambda_i\tau \in \pi_k(BSO(k))$ where $\tau \in \pi_k(BSO(k))$ represents the tangent bundle of S^k. Suppose that the plumbing between any two bundles have the same sing. Let $a_{ii} = 2\lambda_i$. For $i \neq j$ let
$$M_{ij} = \begin{cases} p & \text{if the bundles } i \text{ and } j \text{ are plumbed together } p \text{ times with sign } +1 \\ -p & \text{if the bundles } i \text{ and } j \text{ are plumbed together } p \text{ times with sign } -1 \end{cases}$$

Proposition 3.2.2. The matrix A defines a quadratic form on the free n-dimensional $\mathbb{Z}$-module V. This quadratic form is the same intersection form of the manifold by plumbing the original graph. □

Conversely, given any even quadratic form q on V, we can obtain a graph in the obvious way, and thus, a manifold M with intersection form equal to this quadratic form.

Construction of the manifold

Let $k = 2n$, and choose a ordered basis (indexed by I) of the free $\mathbb{Z}$-module, which has intersection matrix A given at the beginning of this section. For each $i \in I$, set a sphere $S_i = S^{2n}$ and take the disk bundle $D(TS_i) \to S_i$.
Let W be the $4n$-manifold resulting by plumbing the above bundles with respect to the matrix A. Observe that by construction, W is stably parallelizable. The construction of W is equivalent to make a plumbing along the graph



Where each vertex represents the tangent bundle of S^{2n}.

Lemma 3.2.3. The manifold W is $(2n-1)$-connected and ∂W is $(2n-2)$-connected.

Proof. See [Bw2, p.117] □

Theorem 3.2.4. ∂W is a homotopy sphere.

Proof. By the Poincaré-Lefschetz duality, $H_i(W, \partial W, \mathbb{Z}) \cong H^{2k-i}(W, \mathbb{Z})$, so $H_i(W, \partial W, \mathbb{Z}) \cong$Hom$(H_{2k-1}(W, \mathbb{Z}), \mathbb{Z})$. Thus, intersection matrix determines the natural homomorphism

$$H_i(W, \mathbb{Z}) \to H_i(W, \partial W, \mathbb{Z}) \cong \mathrm{Hom}(H_{2k-i}(W), \mathbb{Z})$$

Recall that $H_0(W) \cong \mathbb{Z}$ and $H_i(W) = 0$ for $i \neq 0, 2k$. Consider the exact sequence associated to the pair $(W, \partial W)$.

$$0 \to H_k(\partial W) \xrightarrow{i_*} H_k(W) \xrightarrow{j_*} H_k(W, \partial W) \xrightarrow{\partial} H_{k-1}(\partial W)$$

The map $H_k(W) \to H_k(W, \partial W)$ is given by the intersection matrix A, thus it is an isomorphism and hence $H_k(\partial W) = H_{k-1}(\partial W) = 0$. So ∂W is a homotopy sphere. □

All the results in this section converge to:

Corollary 3.2.5. ∂W is a $(4n-1)$-dimensional exotic sphere. □

3.3 Algebraic Varieties With Singularities

We start recalling some results about the topology of joins. For all the details see [M2].

Definition 3.3.1. Let $A_1, \cdots, A_n$ be topological spaces. We define the *join* of this spaces as the set $A = A_1 * \cdots * A_n$ consisting by elements of the form

$$t_1 a_1 \oplus \cdots \oplus t_n a_n$$

where $t_1, \ldots, t_n \in \mathbb{R}$, $t_i \geq 0$, $t_1 + \cdots + t_n = 1$, and $a_i \in A_i$.

We consider A with the strongest topology such that the functions $t_i : A \to [0, 1]$, $a_i : t_i^{-1}(0, 1] \to A_i$ are continuous.

Theorem 3.3.2. Let A, B, be topological spaces. We have

$$\widetilde{H}_{k+1}(A*B,\mathbb{Z}) \cong \sum_{i+j=k} \widetilde{H}_i(A,\mathbb{Z}) \otimes \widetilde{H}_j(B,\mathbb{Z}) + \sum_{i+j=k-1} \mathrm{Tor}(\widetilde{H}_i(A,\mathbb{Z}), \widetilde{H}_i(B,\mathbb{Z})).$$

Moreover, if B is arcwise connected and A is non vacuous, then $A*B$ is simply connected. □

From the Hurewicz theorem it follows that

Corollary 3.3.3. The join of $(n+1)$ topological spaces is $(n-1)$-connected. □

Now we can construct exotic spheres by considering complex algebraic varieties.
For a sequence of integers $a = (a_1, \dots, a_n)$ with $a_i \geq 2$, let $f(z_1, \dots, z_n) = z_1^{a_1} + \cdots z_n^{a_n}$ be a complex polynomial. Denote by $V(f)$ the set of zeros of f and put $\Sigma(a) = V(f) \cap S^{2n-1}$. Define $\Xi(t) = f^{-1}(t)$ and $V_a = \Xi(1)$.

Remark. Since the derivative of f at a point $z \in \mathbb{C}^n$ is $Df(z) = (a_1 z_1^{a_1-1}, \dots, a_n z_n^{a_n-1})$, the only critical point is 0 and therefore the variety $V(f)$ is an hypersurface with an isolated singularity at 0.

The set $\Sigma(a)$ is a smooth manifold of dimension $2n-3$, since it is embedded as a codimension 2 submanifold of S^{2n-1}.

Let G_{a_j} be the multiplicative cyclic group of order a_j and w_j its respective generator, consider the group $G_a = G_{a_1} \times \cdots \times G_{a_n}$. If we identify each G_{a_j} with the group generated by $\xi_j = e^{\frac{2\pi i}{a_j}}$ the a_j^{th} root of unity. G_a acts over V_a by the action $(w_1^{k_1}, \dots, w_n^{k_n}) \cdot (z_1, \dots, z_n) = (\xi_1^{k_1} z_1, \dots, \xi_n^{k_n} z_n)$.

Lemma 3.3.4. Let $U_a = \{z \in V_a : z_j^{a_j}$ is a non negative real number$\}$. Then U_a is a deformation retract of V_a.

Proof. Consider the complex hyperplanes $X = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sigma z_i = 1\}$ and $S_i = \{z \in X : z_i = 0\}$. Then, there is a retract of the system of hyperplanes $(X, S_1, \dots, S_n)$ over the simplicial system $(\Delta_{n-1}, \partial_1\Delta_{n-1}, \dots, \partial_n\Delta_{n-1})$, illustrated by the figure 3.1

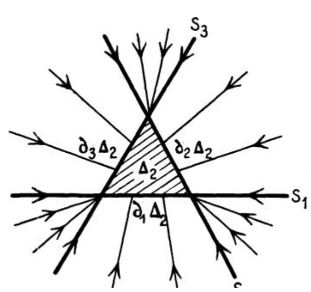


Figure 3.1: Deformation retract of the simplicial system $(\Delta_3, \partial_1\Delta_3, \partial_2\Delta_3, \partial_3\Delta_3)$

Using the change of variables $\xi_i = z_i^{1/a_i}$, we have the retraction of V_a over U_a. □

Note that a element $z \in U_a$ can be identified with elements of the form $z_j = u_j|z_j|$ with $u_j \in G_{a_j}$. Setting $t_j = |z_j|^{a_j}$, then U_a becomes the space of n-tuples

$$t_1 u_1 \oplus \cdots \oplus t_n u_n$$

with $u_j \in G_{a_j}$, $t_j \geq 0$, $\sum_{j=0}^{n} t_j = 1$.

Thus U_a can be identified with the join $G_{a_1} * \cdots * G_{a_n}$ of the finite sets G_{a_j}.
Therefore, from theorem (3.3.2), we get

Proposition 3.3.5. $V_a \subseteq \mathbb{C}^n$ is $(n-2)$-connected, and

$$\widetilde{H}_{n-1}(V_a, \mathbb{Z}) \cong \widetilde{H}_0(G_{a_1}, \mathbb{Z}) \otimes \cdots \otimes \widetilde{H}_0(G_{a_n}, \mathbb{Z}).$$

This is a free abelian group of rank $r = \prod(a_j - 1)$. The other reduced homology groups are zero.

U_a is an $(n-1)$-dimensional simplicial complex which has an $(n-1)$-simplex for each element of G_a. The $(n-1)$-simplex associated to the unit of G_a is denoted by e. All other $(n-1)$-simplices are obtained from e by operations of G_a. Thus we have for the $(n-1)$-dimensional simplicial chain group

$$C_{n-1}(U_a, \mathbb{Z}) = J_a e$$

where J_a is the group ring of G_a. The homology group $\widetilde{H}_{n-1}(U_a) = \widetilde{H}_{n-1}(V_a)$ is an additive subgroup of $J_a e = C_{n-1}(U_a) \cong J_a$.

The face operator ∂_j satisfies $\partial_j = w_j \partial_j$, therefore

$$h = (1 - w_1) \cdots (1 - w_n) e$$

is a cycle. Thus $h \in \widetilde{H}_{n-1}(U_a, \mathbb{Z})$. It follows that $\widetilde{H}_{n-1}(V_a, \mathbb{Z}) = J_a h$.

Theorem 3.3.6. The map $w \mapsto wh$ induces an isomorphism $J_a/I_a \cong \widetilde{H}_n(V_a, \mathbb{Z}) = J_a h$. Where $I_a \subseteq J_a$ is the annihilator ideal of h which is generated by the elements

$$1 + w_j + w_j^2 + \cdots + w_j^{a_j - 1}$$

$(j = 0, \ldots, n)$. Therefore $w_1^{k_1} \cdots w_n^{k_n} h$ is a basis of $\widetilde{H}_{n-1}(V_a, \mathbb{Z})$. □

Now we can compute the cohomology of $\Sigma(a)$ to show that it is homeomorphic to the sphere.
The manifold $Y = \mathbb{C}^n - V(f)$ is a deformation retract of $S^{2n-1} - \Sigma(a)$. The polynomial f induces a fibration[1]

$$V_a \to Y \xrightarrow{f} C^*.$$

By the long exact sequence of homotopy associated to this fibration we get $\pi_1(Y) \cong \mathbb{Z}$, $\pi_{n-1}(Y) = J_a/I_a$. Computing the homology of Y associated to the spectral sequence of this fibration with E^2-term, the homology with local coefficients

$$E^2_{p,q} = H_q(C^*, H_q(V_a, \mathbb{Z})).$$

For each $t = e^{2\pi i\theta} \in S^1$ who describes a circle, there exist an isotopy $f_\theta : \Xi(1) \to \xi(t)$ defined by $f_\theta(\xi_i) = e^{\frac{2\pi i\theta}{a_j}} \xi_i$. Particularly, if $t = 1$, we obtain the automorphism $w = w_1 \cdots w_n$.

So from [DK] we get that $E^\infty_{p,q} = E^2_{p,q}$, and the only no trivial groups are $E^2_{0,0} \cong E^2_{1,0} \cong \mathbb{Z}$ and

$$E^2_{0,n-1} \cong \operatorname{coker}(1 - w)$$

[1] It is known as the Milnor fibration theorem

$$E^2_{1,n-1} \cong \ker(1-w).$$

It follows that $H_i(Y,\mathbb{Z}) = 0$ for $i \neq 0,1,n-1,n$. In the case $i = n-1$, $H_i(Y,\mathbb{Z}) = 0$ if and only if $1-w : J_a/I_a \to J_a/I_a$ is an isomorphism, that is equivalent to the condition that

$$\det(1-w) = \pm 1.$$

Lemma 3.3.7. Define $\Delta_a(t) = \det(t \cdot 1 - w)$. Then

$$\Delta_a(t) = \prod_{0<i_k<a_k} (t - \xi_1^{i_1} \cdots \xi_n^{i_n}).$$

Proof. Consider J_a/I_a as the Tensor product $\bigotimes_{k=1}^n V_k$ where V_k is the $\mathbb{Z}$-module spanned by $\{w_k^i\}$. Thus $w : J_a/I_a \to J_a/I_a$ can be seen as the map $w_1 \otimes \cdots \otimes w_n$ where $w_k : V_k \to V_k$ is the multiplication by w_k.

Tensoring everything with $\mathbb{C}$, we find that for each a_k^{th} root of unity, $x_k = \xi_k^{i_k}$ the vector $1 + x_k w_k + \cdots + (x_k w_k)^{a_k - 1}$ is an eigenvector of w_k with eigenvalue x_k^{-1}. Therefore the eigenvalues of w are all the numbers $x_1^{-1} \cdots x_n^{-1}$. So

$$\Delta_a(t) = \prod_{0<i_k<a_k} (t - \xi_1^{i_1} \cdots \xi_n^{i_n}).$$

□

In particular, $\Delta_a(1)$ has positive real part. So we have the following result.

Theorem 3.3.8. $\Sigma(a)$ is a homology sphere of dimension $(2n-3)$ for $n \geq 4$ if and only if $\Delta(1) = 1$.

Proof. Notice that $\Sigma(a)$ is a deformation retract of $V(f) - \{0\}$. Consider the $2-$codimensional manifold $X = \{z \in V(f) : z_n = 0\}$. There is a surjection $\pi_1(V(f) - X_a) \to \pi_1(V(f) - \{0\})$ and a fibration $V_{\widetilde{a}} \to V(f) - X_a \to C^*$, where $\widetilde{a} = (a_1, \ldots, a_{n-1})$. Thus $\pi_1(V(f) - X_a)$ and so $\pi_1(V(f) - \{0\})$ is abelian.

Therefore, from the Alexander Duality:[1]

$$\pi_1(\Sigma(a)) \cong H_1(\Sigma_a, \mathbb{Z}) \cong H^{2n-3}(S^{2n-1} - \Sigma(a), \mathbb{Z}) \cong H^{2n-3}(Y, \mathbb{Z}) = 0.$$

And using the Hurewicz theorem for $i \leq n-3$

$$\pi_i(\Sigma(a)) \cong H_i(\Sigma_a, \mathbb{Z}) \cong H^{2n-2-i}(S^{2n-1} - \Sigma(a), \mathbb{Z}) \cong H^{2n-2-i}(Y, \mathbb{Z}) = 0.$$

In the case of $i = n-2, n-1$, we use the Alexander duality and that $H^{n-1}(Y,\mathbb{Z}) = H^n(Y,\mathbb{Z}) = 0$ if and only if $\Delta_a(1) = 1$ (Observe that we avoid the case $\Delta_a(1) = -1$ using lemma (3.3.7)). □

So, by the h-Cobordism Theorem, we get that actually $\Sigma(a)$ is homeomorphic to S^{2n-3}. In the following results we will compute the signature of this manifold, and in general, this manifold is not diffeomorphic to the sphere.

Recall that V_a is a $(2n-2)$-dimensional oriented manifold without boundary, therefore there is a bilinear intersection form well defined over $H_{n-1}(V_a, \mathbb{Z})$.

[1] **Alexander Duality**: Let $X \subseteq S^n$ be a compact and locally contractile space. Define $Y = S^n - X$. Then there is an isomorphism $H_q(Y, \mathbb{Z}) \cong H^{n-q-1}(X, \mathbb{Z})$

Theorem 3.3.9. The signature of V_a, is given by

$$\sigma(V_a) = \sum_{0<j_k<a_k} (-1)^{[j_1/a_1+\cdots+j_n/a_n]},$$

where $j = (j_1, \ldots, j_n)$ is a $n-$tuple of integers such that $j_k < a_k$ and $n \geq 5$ odd.

Proof. Let n odd. As a basis of $H_{n-1}(V_a, \mathbb{Z}) = J_a/I_a \otimes \mathbb{C}$ use the eigenvectors introduced in the proof of lemma (3.3.7), namely

$$v_j = \prod_{k=1}^{n}(1 + x_k w_k + \cdots + (x_k w_k)^{a_k-1}),$$

where $x_k = e^{2\pi i j_k/a_k}$ and $j = (j_1, \cdots, j_n)$ is an $n-$tuple of integers with $0 < j_k < a_k$.
By explicit calculation of the intersection numbers, we get

$$\langle v_j, v_i\rangle = (-1)^{(n-1)(n-2)/2}(1 - x_1 \cdots x_n) \prod_{k=1}^{n}(1 - x_k^{-1})(1 + x_k y_k + \cdots + (x_k y_k)^{a_k-1}). \tag{3.1}$$

This is 0 unless $i_k + j_k = a_k$ for all k. Therefore the vectors $v_j + v_{a-j}$ and $i(v_j - v_{a-j})$ form a basis of $J_a/I_a \otimes \mathbb{C}$, with respect to which the intersection matrix is diagonal.
These entries are given by

$$\langle v_j + v_{a-j}, v_j + v_{a-j}\rangle = \langle i(v_j - v_{a-j}), i(v_j - v_{a-j})\rangle = 2\langle v_j, v_{a-j}\rangle$$

which are real numbers. Using formula (1) we get then

$$\begin{aligned}
\langle v_j, v_{a-j}\rangle &= (-1)^{(n-1)/2}(1 - x_1 \cdots x_n) \prod_k (1 - x_k^{-1}) a_k \\
&= (-1)^{(n-1)/2}(\prod_k a_k)(\prod_k (1 - x_k^{-1}) + \prod_k (1 - x_k)) \\
&= 2\mathrm{Re}(-1)^{(n-1)/2}(\prod_k a_k)(\prod_k 1 - x_k^{-1}) \\
&= 2\mathrm{Re}(-1)^{(n-1)/2}(\prod_k a_k)(-2ie^{\pi i j_k/a_k} \sin(\pi \frac{j_k}{a_k})) \\
&= \Big(\prod_k 4a_k \sin(\pi \frac{j_k}{a_k})\Big)\mathrm{Re}(-e^{\pi i(\frac{1}{2} + \sum_k \frac{j_k}{a_k})}).
\end{aligned}$$

Since $\sin(\pi \frac{j_k}{a_k}) > 0$ because $0 < j_k < a_k$, then the above expression is positive exactly when

$$\mathrm{Re}(-e^{\pi i(\frac{1}{2} + \sum_k \frac{j_k}{a_k})}) < 0,$$

or equivalently,

$$2l < \sum_k \frac{j_k}{a_k} < 1 + 2l.$$

Thus, the signature of V_a is equal to $\tau_+ - \tau_-$ where $\tau_\pm$ is the number of n-tuples $j = (j_1, \ldots, j_k)$ such that $0 < j_k < a_k$ and $\sum_{k=1}^{n} \frac{j_k}{a_k}$ mod 2 lies between 0 and ± 1. This means that

$$\sigma(V_a) = \sum_{0<j_k<a_k} (-1)^{[j_1/a_1+\cdots+j_n/a_n]}.$$

□

A careful computation gives:

Corollary 3.3.10. For $a = (a_1, \dots, a_n) = (3, 6k-1, 2, \dots, 2)$ with n odd and k any integer.

$$\sigma(V_a) = (-1)^{(n+1)/2} 8k.$$

□

Define $M_a(\varepsilon) = \Xi(\varepsilon) \cap D^{2n}$ and $\Sigma_a(\varepsilon) = \Xi(\varepsilon) \cap S^{2n-1}$ for $\varepsilon > 0$ small. So $\Sigma(a)$ is diffeomorphic to $\Sigma_a(\varepsilon)$ and it is the boundary of $M_a(\varepsilon)$. The interior of M_a is diffeomorphic to $\Xi(\varepsilon) \cong V_a$. Since the normal bundle of M_a is trivial, M_a is stably parallelizable.

Corollary 3.3.11. The manifold $\Sigma(3, 6k-1, 2, \dots, 2)$ is a exotic $(4n-1)$-dimensional sphere.

Later we will use the following results, for details see Appendix B and [Bk].

Theorem 3.3.12. For the sequence $a = (a_1, \dots, a_{n+1}) = (d, 2, \dots, 2)$, the $(2n-1)$-manifold $\Sigma(a)$ is diffeomorphic to the plumbing of $d-1$ copies of the tangent bundle of S^n along the graph □


d − 1 *vertices.*

Theorem 3.3.13. The $(4n-1)$-manifold constructed in section (3.2) is diffeomorphic to the manifold $\Sigma(3, 5, 2, \dots, 2)$. □

Theorem 3.3.14. Consider the sequence $a = (d, 2, \dots, 2)$ and set M_a as above. Then the Arf-Kervaire invariant of M_a is equal to

$$c(M_a) = \begin{cases} 0 & \text{if } d = \pm 1 \bmod 8 \\ 1 & \text{if } d = \pm 3 \bmod 8 \end{cases}$$

□

Corollary 3.3.15. $\Sigma(3, 2, \dots, 2)$ is a exotic $(2n-1)$-dimensional sphere. □

Chapter 4

Groups of Homotopy Spheres

4.1 Construction of the Group Θ_n

All manifolds in this section are assumed to be compact, oriented and differentiable.

Definition 4.1.1. Let M, N be two closed oriented n-dimensional manifolds, we say that M and N are *h-cobordant* if the disjoint sum $M + (-N)$ is the boundary of some manifold W and both M and $-N$ are deformation retracts of W. Notice that $[M] = [N]$ as elements of the group Ω_n^{SO}.

Remark. Observe that if M is diffeomorphic to N, the oriented manifold $M \times [0, 1]$ has as boundary $M + (-M) \cong M + (-N)$

Recall the construction of the connected sum $M \# N$ of two n-manifolds M and N. Choose imbeddings $i_1 : D^n \to M$, $i_2 : D^n \to N$. Obtain $M \# N$ from the disjoint sum

$$(M - i_1(0)) + (N - i_2(0))$$

by identifying $i_1(tu)$ with $i_2((1 - t)u)$ for each unit vector $u \in S^{n-1}$ and $0 < t < 1$. Choose the orientation for $M \# N$ which is compatible with that of M and N. This is possible since the correspondence $i_1(tu) \mapsto i_2((1 - t)u)$ preserves orientation.

Proposition 4.1.2. The manifolds $M \# N$ and $M + N$ are h-cobordant.

Proof. Consider the cylinder $M \times I$ and let $i_1 : D^{n+1} \to M \times I$ such that $i_1(0) = (m, 1)$ for some $m \in M$. Similarly choose $i_2 : D^{n+1} \to N \times I$ with $i_2(0) = (n, 1)$. Now make an identification

$$((M \times I) - i_1(0)) + ((N \times I) - i_2(0))$$

as above. This construction gives a manifold W with boundary $(M \# N) + (-(M + N))$. □

As an immediate result of this proposition, we get that the Cobordism groups can be defined with the connect sum as operation addition, instead of the disjoint union.

Lemma 4.1.3. The connected sum operation is well defined (does not depend of the choosing of the imbeddings), associative, commutative up to orientation preserving diffeomorphism. The sphere S^n serves as identity element.

Proof. Let $i_1, i_1' : D^n \to M$ imbeddings. The map $x \mapsto (i_1')^{-1} i_1(x)$ defines a diffeomorphism of D^n onto itself which is a orientation preserving map. There is a diffeomorphism $f : M \to M$ such that $i_1'(t) = f(i_1(t))$ if $t \in D^n$. Given two embeddings $i_2, i_2' : D^n \to N$ we also construct a diffeomorphism $g : N \to N$ with $i_2'(t) = f(i_2(t))$. Denote by $M \# N$ the connected sum associated to i_1 and i_2 and $(M \# N)'$ the connected sum associated to i_1' and i_2'. The above constructed diffeomorphism induces a diffeomorphism

$$H : M_1 \# M_2 \to (M_1 \# M_2)'.$$

The associativity and commutativity is immediate from the definition.
The manifold $M \# S^n$ is diffeomorphic to M since $S^n - i(D^n)$ is diffeomorphic to $int(D^n)$. □

Lemma 4.1.4. Let M, M' and N be closed and simply connected $n-$manifolds with $n \geq 3$. If M is h-cobordant to M' then $M \# N$ is h-cobordant to $M' \# N$.

Proof. Let W a manifold with $\partial W = M + (-M')$, where M and $-M'$ are deformation retracts of W. Let A a curve in W form a point $p \in M$ to a point $p' \in M'$ with a tubular neighborhood diffeomorphic to $\mathbb{R}^n \times [0,1]$. So there is an imbedding

$$i : \mathbb{R}^n \times [0,1] \to W$$

with $i(\mathbb{R}^n \times 0) \subseteq M$, $i(\mathbb{R} \times 1) \subseteq M'$ and $i(0 \times [0,1]) = A$.

Consider the manifold $Z = (W - A) + (N - i_2(0)) \times [0,1]$ by identifying $i(tu, s)$ with $i_2((1-t)u) \times s$ for each $0 < t < 1$, $0 \leq s \leq 1$, $u \in S^{n-1}$. Z is a compact manifold with boundary $M \# N + (-(M' \# N))$.

Let see that these both boundaries are deformation retracts of Z. Consider the inclusion map

$$M - p \xrightarrow{j} W - A$$

since $n \geq 3$, both of these manifolds are simply connected. On the other hand, the homology exact sequence of the pair $(M, M-p)$ and $(W, W-A)$ shows that j induces isomorphism of homology groups. Hence a homotopy equivalence. Using this and the Mayer-Vietoris sequence over the manifolds $M \# N$ and Z we get that

$$H_i(M \# N, \mathbb{Z}) \cong H_i(M-p, \mathbb{Z}) \oplus H_i(N-q, \mathbb{Z}) \cong H_i(W-A, \mathbb{Z}) \oplus H_i(N - i_2(0) \times [0,1], \mathbb{Z}) \cong H_i(Z, \mathbb{Z})$$

So the inclusion $M \# N \to Z$ is a homotopy equivalence. Thus $M \# N$ is a deformation retract of Z.

Similarly, $M' \# N$ is a deformation retract of Z. □

Lemma 4.1.5. A simply connected manifold M is h-cobordant to the sphere S^n if and only if M bounds a contractible manifold.

Proof. Suppose that $M + (-S^n) = \partial W$. Fill the disk D^{n+1} inside S^n to obtain a manifold W' with $\partial W = M$. Since S^n is a deformation retract of W, then it follows that W' is contractible.

Conversely, if $M = \partial W'$ with W' contractible, let $D^{n-1} \hookrightarrow W'$ a local chart and remove its interior to get a simply connected manifold W, with $\partial W = M + (-S^n)$. Mapping the homology exact sequence of the pair (D^{n+1}, S^n) into the pair (W', W) the inclusion $S^n \to W$ induces a homology isomorphism we get that the inclusion $S^n \to W$ induces a homology isomorphism, hence S^n is a deformation retract of W. Applying the Poincaré-Lefschetz duality

$$H_k(W, M) \cong H^{n+1-k}(W, S^n).$$

This proves that the inclusion $M \hookrightarrow W$ induces isomorphism of homology groups. Since M is simply connected, M is a deformation retract of W. □

Lemma 4.1.6. If M is a homotopy sphere, then $M\#(-M)$ bounds a contractible manifold.

Proof. Let $H^2 \subseteq D^2$ denote the half-disk consisting of all $(t\sin(\theta), t\cos(\theta))$ with $0 \leq t \leq 1$, $0 \leq \theta \leq \pi$, and let $\frac{1}{2}D^n \subseteq D^n$ denote disk of radius $\frac{1}{2}$. Given an imbedding $i : D^n \to M$, construct a manifold W from the disjoint union

$$(M - i(\frac{1}{2}D^n)) \times [0, \pi] + S^{n-1} \times H^2$$

by identifying $i(tu) \times \theta$ with $u \times ((2t-1)\sin\theta, (2t-1)\cos\theta)$ for each $\frac{1}{2} < t \leq 1$, $0 \leq \theta \leq \pi$.

Therefore W is a differentiable manifold with boundary $\partial W = M\#(-M)$. Moreover, W contains $M - Int(i(\frac{1}{2}D^n))$ as a deformation retract and therefore is contractible. □

Theorem 4.1.7. Let Θ_n denote the set of all h-cobordism classes of homotopy $n-$spheres. Then Θ_n is an abelian group under the connected sum operation.

Proof. By lemmas (4.1.3) and (4.1.4), the connected sum is a well defined, associative, commutative operation. The class of the sphere S^n is the zero element. And by lemmas (4.1.5) and (4.1.6) each element of Θ_n has an inverse. □

This group is called the n^{th}-*homotopy sphere cobordism group*, and in the following lines we investigate the structure of this group.

Remark. By the h-cobordism theorem, for $n \geq 5$, study the structure of this group is equivalent to study the group of classes of homotopy spheres under the relation of diffeomorphism. So, the number of n-dimensional exotic spheres up to diffeomorphism is equal to the cardinal of the group Θ_n.

Since in lower dimensions the differentiable structure of a manifold is completely determined by its topology , we get:

Proposition 4.1.8. For $n = 1, 2, 3$ the group Θ_n is trivial.

Proof. In the case $n = 1$, an element $[M] \in \Theta_1$ is homeomorphic to S^1 and it is known from differential topology that the only connected and compact smooth 1-manifolds are (up to diffeomorphism) $[0, 1]$ and S^1. So M is diffeomorphic to S^1.

For the case $n = 2$, all oriented compact 2-dimensional manifolds are completely determined up to diffeomorphism by the Euler characteristic, that is, a 2-manifold M is diffeomorphic to S^2 or to a connected sum of T^2's. So if M is a 2-homotopy sphere it is diffeomorphic to S^2.

The case $n = 3$, from [Mo], for every $3-$dimensional manifold M there exist an unique differentiable structure over M. Then if M is homeomorphic to S^3 it is necessarily diffeomorphic to the sphere. □

Proposition 4.1.9. For $n \geq 4$, the group Θ_{2n-1} is non trivial. $|\Theta_7| \geq 4$.

Proof. From (3.1.12), (3.2.5), (3.3.11) and (3.3.15). □

4.2 Construction of the Subgroup bP_{n+1}

Let (M, f) be a closed framed manifold M of dimension n. By Whitney's embedding theorem there is an embedding $i : M \to \mathbb{R}^{n+k}$ for $k > n$. The Thom–Pontryagin isomorphism with the (B, f)-structure associated to a normal framed manifolds works in the following way to obtain a map $S^k \to S^{n+k}$.(Refer to theorem (A.19))

A framing of $TM \oplus \epsilon^1$ induces a framing φ of $\nu(i)$, since $TM \oplus \nu(i)$ is trivial and using (A.13) over the Whitney sum $(TM \oplus \epsilon^1) \oplus \nu(i)$. By the tubular neighborhood theorem, set N a tubular neighborhood in $\mathbb{R}^{n+k}$ around M which is diffeomorphic to the normal bundle of M. Consider a map from $\mathbb{R}^{n+k}$ to $S^k = \mathbb{R}^k \cup \{\infty\}$ which sends the complement of N to ∞ and sends each normal fiber to $\mathbb{R}^k$ using the trivialization of that fiber. The maps extends to S^{n+k} by sending ∞ to ∞.

Thus we have a well defined element

$$p(M, f) : S^{n+k} \to S^k \in \Pi_n = \lim_{k\to\infty} \pi_{n+k}(S^k).$$

Allowing the trivialization φ to vary, we obtain a set of elements

$$p(M) = \{p(M, \varphi)\} \subseteq \Pi_n.$$

Lemma 4.2.1. The subset $p(M)$ contains the zero element of Π_n if and only if M bounds a parallelizable manifold.

Proof. By the Thom–Pontryagin theorem,

$$p(M, \varphi) \sim 0 \text{ iff } [M] \equiv 0 \in \Omega_n^{fr} \text{ iff } M = \partial W$$

for some framed manifold W. Since for compact manifolds with boundary the concept of be parallelizable and stably parallelizable are equivalent, we are done. □

Lemma 4.2.2. If M is h-cobordant to N, then $p(M) = p(N)$.

Proof. Let W be a manifold such that $M + (-N) = \partial W$, and by Whitney's embedding theorem, choose a embedding of W in $S^{n+k} \times [0, 1]$ so that $M \hookrightarrow S^{n+k} \times 0$ and $N \hookrightarrow S^{n+k} \times 1$. Let φ a framing of the normal bundle of M, which extends to a framing ψ of the normal bundle of W since M is a retract of W. The restriction $\varphi' = \psi|_N$ gives a framing of the normal bundle of N. Therefore, (W, ϕ) gives a homotopy between $p(M, \varphi')$ and $p(N, \varphi')$. This construction could be made starting with a normal framing of N, so we get that $p(M) = p(N)$. □

Lemma 4.2.3. If M and N are stably parallelizable, then $p(M) + p(N) \subseteq p(M\#N)$

Proof. Let $W_1 = M \times [0, 1]$ and $W_2 = N \times [0, 1]$, and set $B_1 = M \times 1$ and $B_2 = N \times 0$. Let $H^{n+1} = \{x = (x_0, x_1, \ldots, x_n) : |x| = 1, x_0 \leq 0\}$ and $D^n \subseteq H^{n+1}$ the subset $x_0 = 0$. Choose embeddings

$$i_k : (H^{n+1}, D^n) \to (W_k, B_k)$$

so that $i_2 \circ i_1^{-1}$ reverses the orientation. Denote W the manifold $(W_1 - i_1(0)) + (W_2 - i_2(0))$, by identifying $i_1(tu)$ with $i_2((1-t)u)$ for $0 < t < 1$ and $u \in S^n \cap H^{n+1}$. Thus, W is a differentiable manifold with

$\partial W = M\#N + (-M) + (-N)$ and M has the homotopy type $M \vee N$.

Choose an embedding $W \to S^{n+k} \times [0,1]$, with a k large so that $W \cap (S^{n+k} \times 0) = (-M) + (-N)$, and $W \cap (S^{n+k} \times 1) = M\#N$. So given two k-frames φ_1 and φ_2 on $(-M)$ and $(-N)$ respectively, we can extend both them to a k-frames throughout W. Denote ψ the restriction of this framing to $M\#N$. So we get an homotopy $p(M, \varphi_1) + p(N, \varphi_2)$ and $p(M\#N, \psi)$. □

Since any homotopy sphere is stably parallelizable (see [KM, p.508]) we have,

Theorem 4.2.4. The set $p(S^n) \subseteq \Pi_n$ is a subgroup of the stable homotopy group Π_n. For any homotopy sphere Σ, the set $p(\Sigma)$ is a coset of this subgroup. Thus the correspondence $\Sigma \mapsto p(\Sigma)$ defines a homomorphism p from Θ_n to the quotient group $\Pi_n/p(S^n)$.

Proof. Using the lemma (4.2.3) together the identities

- $S^n\#S^n \cong S^n$ implies $p(S^n) + p(S^n) \subseteq p(S^n)$ ($p(S^n)$ is a subgroup).
- $S^n\#\Sigma \cong \Sigma$ implies $p(S^n) + p(\Sigma) \subseteq p(\Sigma)$ ($p(\Sigma)$ is a union of cosets of this subgroup).
- $\Sigma\#(-\Sigma) \sim S^n$ implies $p(\Sigma) + p(-\Sigma) = p(S^n)$ ($p(\Sigma)$ is a single coset).

□

Definition 4.2.5. By lemmas (4.2.2) and (4.2.3), the kernel of $p : \Theta_n \to \Pi_n/p(S^n)$ consists exactly of all h-cobordism classes of homotopy n-spheres which bound parallelizable manifolds. Thus these elements form a group which we denote by $bP_{n+1} \subseteq \Theta_n$.

Theorem 4.2.6. The group Θ_n/bP_{n+1} is finite.

Proof. By theorem (4.2.4), Θ_n/bP_{n+1} is isomorphic to a subgroup of $\Pi_n/p(S^n)$. Since the groups Π_n are finite (1.2.2), we get the result. □

In other words, *The number of exotic spheres that do not bound parallelizable manifolds is finite.*

4.3 Some Computations on bP_{n+1}

In this section we will use the theory of *Spherical Modifications* introduced in Appendix A, which is a powerful tool to study the homotopy type of the manifolds through a certain kind of "surgery". The important fact of this technique is that it is invariant under the boundary.

The group bP_{n+1}, n odd

Theorem 4.3.1. Let n be an odd integer. Then $bP_{n+1} = 0$.

Proof. Let M^{n+1} a compact framed manifold such that ∂M is a homotopy sphere. By theorem (A.20) and Poincaré duality, M is χ-equivalent to a contractible manifold. □

The groups bP_6 and bP_{14}

Lemma 4.3.2. Let M be a $(k-1)$-connected manifold of dimension $2k$, $k \geq 3$. Suppose that $H_k(M, \mathbb{Z})$ is free abelian with a basis $\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\}$ where

$$\langle \lambda_i, \lambda_j \rangle = 0, \ \ \langle \lambda_i, \mu_j \rangle = \delta_{ij}$$

for all i, j. Suppose further that λ_i can be represented by disjoint embedded spheres with trivial normal bundles. Then M is χ-equivalent to a contractible manifold.

Proof. Let $\varphi_0 : S^k \to M$ be an embedding that represents the homology class λ_m. Since the normal bundle is trivial, φ_0 can be extended to an embedding $\varphi : S^k \times D^k \to M$. Let $M' = \chi(M, \varphi)$ and $M_0 = M - \mathrm{int}(\varphi(S^k \times D^k))$.

Consider the exact sequences

$$0 \to H_k(M_0, \mathbb{Z}) \to H_k(M, \mathbb{Z}) \xrightarrow{i_*} H_k(M, M_0, \mathbb{Z}) \xrightarrow{\partial}$$

$$H_{k+1}(M', M_0, \mathbb{Z}) \xrightarrow{\partial} H_k(M_0, \mathbb{Z}) \to H_k(M') \to 0$$

which by excision shows that $H_k(M, M_0)$ is infinite cyclic and thus there is a diagram

$$\begin{array}{ccccccccccc} & & \mathbb{Z} & & & & & & & & \\ & & \downarrow & \searrow^{\lambda_m} & & & & & & & \\ 0 & \longrightarrow & H_k(M_0, \mathbb{Z}) & \longrightarrow & H_k(M, \mathbb{Z}) & \xrightarrow{\langle \cdot, \lambda_m \rangle} & \mathbb{Z} & \longrightarrow & H_{k-1}(M_0, \mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow & & & & & & & & \\ & & H_k(M', \mathbb{Z}) & & & & & & & & \\ & & \downarrow & & & & & & & & \\ & & 0 & & & & & & & & \end{array}$$

Since $\langle \mu_m, \lambda_m \rangle = 1$ it follows that $H_{k-1}(M_0, \mathbb{Z}) = 0$. From this we get the fact that M_0 and M' are $(k-1)$-connected. The group $H_k(M_0, \mathbb{Z})$ is isomorphic to the subgroup of $H_k(M, \mathbb{Z})$ generated by $\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_{m-1}\}$. The group $H_k(M', \mathbb{Z})$ is isomorphic to a quotient group of $H_k(M_0, \mathbb{Z})$. It has a basis $\{\lambda'_1, \ldots, \lambda'_{m-1}, \mu'_1, \mu'_{m-1}\}$ where each λ'_i corresponds to a coset $\lambda_i + \lambda_m\mathbb{Z} \subseteq H_k(M, \mathbb{Z})$. And respectively with the μ'_j.

The manifold M also satisfies the hypothesis of this theorem, we only have to verify that

$$\langle \lambda'_i, \lambda'_j \rangle = 0, \ \ \langle \lambda'_i, \mu'_j \rangle = \delta_{ij}.$$

Each λ'_i or μ'_j can be represented by a sphere embedded in M_0 and representing the homology class λ_i or μ_j of M. Thus the intersection numbers in M' are the same as those in M.

Iterating this construction m times, the result will be a k-connected manifold.

□

Theorem 4.3.3. The groups bP_6, bP_{14} are both zero.

Proof. Let M be a stably parallelizable manifold of dimension $2k$ such that ∂M is a homology sphere. By theorem (A.20) we can assume that M is $(k-1)$ connected, by Poincaré duality it follows that $H_k(M,\mathbb{Z})$ is free abelian and the intersection matrix has determinant ± 1.

Case $k = 3, 7$. Since k is odd the intersection matrix is skew symmetric, hence there exists a basis for $H_k(M,\mathbb{Z})$, namely $\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\}$ with

$$\langle \lambda_i, \lambda_j \rangle = \langle \mu_i, \mu_j \rangle = 0, \quad \langle \lambda_i, \mu_j \rangle = \delta_{ij}$$

The obstruction to framing any embedded sphere $S^k \to M$ lies in $\pi_{k-1}(SO(k))$ which is equal to 0 for $k = 3, 7$. So by lemma (4.3.2) we have that M is χ-equivalent to a contractible manifold. $\square$

The group bP_{4n}

Theorem 4.3.4. Let (M^{4n}, f) be a compact framed $(2n-1)$-connected manifold with ∂M a homotopy sphere. Then (M, f) is χ-equivalent into a contractible manifold if and only if $\sigma(M) = 0$.

Proof. One direction follows from theorem (A.10) since σ is an invariant under cobordism.

Conversely, suppose that $\sigma(M) = 0$, by (A.15), one can suppose that M is $(n-1)$-connected. Since $\sigma(M) = 0$, there is a basis $\{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\}$. Each λ_i can be represented by an embedding $f_i : S^{2n} \to M^{4n}$. Since $\langle \lambda_i, \lambda_j \rangle = 0$, the f_i can be chosen to be disjoint. Let $\nu(f_i)$ be the normal bundle associated to the embedding f_i, then the obstruction $[\nu(f_i)] \in \pi_{2n-1}(SO(2n))$.

Recall that $TS^{2n} \oplus \nu(f_i) \cong f_i^*(TM)$, since TM and TS^{2n} are stably trivial, so is $\nu(f_i)$, that is $i_*[\nu(f_i)] = 0 \in \pi_{2n-1}(SO(2n+1))$ where $i : SO(2n) \to SO(2n+1)$ is the standard inclusion. We have a fibration, $SO(n-1) \xrightarrow{i_n} SO(n) \xrightarrow{p_n} S^n$ which induces the following diagram in homotopy groups,

$$\begin{array}{ccccc} \pi_{2n}(S^{2n}) & \xrightarrow{d_{2n}} & \pi_{2n-1}(SO(2n)) & \xrightarrow{(i_{2n-1})_*} & \pi_{2n-1}(SO(2n+1)) \\ & \searrow^{\times 2} & \downarrow{\scriptstyle (p_{2n-1})_*} & & \\ & & \pi_{2n-1}(S^{2n-1}) & & \end{array}$$

A computation gives

$$(p_{2n-1})_*[\nu(f_i)] = \langle \lambda_i, \lambda_i \rangle [S^{2n-1}] = 0$$

$$[\nu(f_i)] \in ker((i_{2n-1})_*) = im(d_{2n})$$

$$[\nu(f_i)] \in Im(d_{2n}) \cap ker((p_{2n-1})_*) = 0.$$

And since $(p_{2k-1})_* d_{2k}$ is multiplication by 2, we get that $[v(f_i)] = 0$, or equivalently, $\nu(f_i)$ is trivial.

Thus, by lemma (4.3.2), M is χ-equivalent to a contractible manifold. □

By the h-cobordism theorem follows

Corollary 4.3.5. Let Σ be a homotopy sphere which bounds a stably parallelizable $4n$-manifold M. $\sigma(M) = 0$ if and only if Σ is diffeomorphic to S^{4n-1}. □

Theorem 4.3.6. Let $n > 1$ and $t \in \mathbb{Z}$. There exists a framed $4n$ manifold (M, f) with ∂M a homotopy sphere and $\sigma(M) = 8t$.

Proof. Let W be the manifold constructed in section 4.3. Set $M = W \# \cdots \# W$ (t-times). □

Definition 4.3.7. Let $b_n : \mathbb{Z} \to bP_{4n}$ be the group homomorphism defined by $b_n(t) = [\partial W]$ where W is the framed manifold with signature $8t$ and ∂W is a homotopy sphere.

Lemma 4.3.8. b_n is well defined, that is, if W_1 and W_2 are as above, then ∂W_1 is $h-$cobordant to ∂W_2. Furthermore, b_n is surjective.

Proof. b_n is surjective immediately by theorem (4.3.6). In order to prove that b_k is well defined, it suffices to show that the connected sum $\partial W_1 \# \partial W_2'$ is cobordant to $\emptyset$. Set $W = W_1 \# (-W_2)$, then $\partial W = \partial W_1 \# \partial W_2$. Therefore, $\sigma(W) = \sigma(W_1) - \sigma(W_2) = 0$, so by theorem (4.3.4) W is χ-equivalent to a contractible manifold, that is $\partial W_1 \# \partial W_2' \equiv \emptyset$. □

Corollary 4.3.9. There is an isomorphism of groups, $bP_{4n} \cong \mathbb{Z}/ker(b_n)$. □

To compute the group bP_{4n}, we try to determine $ker(b_n)$.

Definition 4.3.10. An *almost framed manifold* is pair (M, f) where f is a framing of $TM|_{M-\{x\}}$ for some $x \in M$.

Proposition 4.3.11. $t \in ker(b_n)$ if and only if there exists an almost framed closed $4n$-manifold with signature $8t$.

Proof. Suppose that $t \in ker(b_n)$. Then there is a framed manifold (M, f) with signature $8t$, whose boundary Σ is a homotopy sphere that bounds a contractible manifold D. Set $N = D + M$ identifying the common boundary Σ. Thus $\sigma(N) = 8t$ and $TN|_{N-\{x\}}$ is parallelizable for any $x \in N$.

Conversely, if N is an almost framed $4n$-manifold with $\sigma(N) = 8t$, let $D^{4n} \subseteq N$ be any embedded disc which contains the point $x \in N$. Then $N-\text{int}(D^{4n})$ is a framed manifold with signature $8t$ and $\partial N \cong S^{4n-1}$. □

Recall that for any closed manifold M^{4n}, the Hirzebruch signature theorem states that

$$\sigma(M) = \langle L_n(p_1(M), \ldots, p_n(M)), \mu_M \rangle$$

where L_n is a rational function and $p_i(M)$ are the Pontryagin classes of M. Here we use that

$$L_n(x_1, \ldots, x_n) = s_n x_n + R(x_1, \ldots, x_{n-1})$$

and

$$s_n = \frac{2^{2n}(2^{2n-1} - 1)B_n}{(2n)!},$$

where B_n is the n^{th} Bernoulli number.

Therefore, if (M, f) is an almost framed closed manifold. Since $p_i(M) = 0$ for $i < n$, $\sigma(M) = s_n p_k(n)$. Define $\sigma_{M,f} \in \pi_{4n-1}(SO(4n+1) \cong \mathbb{Z})$ the obstruction to extending the almost framing f to a framing of the bundle $TM \oplus \epsilon^1$. Let $x \in M$ be the point where f is not defined. Set D^{4n} a neighborhood of x and let f' be the usual framing of D^{4n}. So, $\sigma_{M,f}$ is the obstruction to the stable framings f and f' agree in $D^{4n} - \{x\} \cong S^{4n-1}$.

Now let $\tau : M \to BSO(4n+1)$ be the classifying map of $TM \oplus \epsilon^1$. Since $M - \{x\}$ is parallelizable, $\tau|_{M-\{x\}}$ is null homotopic and thus factors as

$$\begin{array}{ccc} M & \xrightarrow{\tau} & BSO(4n+1) \\ \downarrow{\scriptstyle \phi} & \nearrow & \\ S^{4n} & & \end{array}$$

Where ϕ maps to a point the complement of int D^{4n}. So there is a $4n$-stable bundle η over S^{4n} with $\phi^*\eta \cong TM \oplus \epsilon^1$, and therefore $[\eta] = \pm\sigma_{M,f}$. (Here we use that in general the set k-plane bundles over S^n (up to isomorphism) is in one-to-one correspondence with $\pi_{n-1}(SO(k))$).

Theorem 4.3.12. If η is an stable vector bundle over S^{4n}, $p_n(\eta) = \pm a_n(2n-1)![\eta]$ where $a_{2m+1} = 1$ and $a_{2m} = 2$.

Proof. By definition $p_n(\eta) = c_{2n}(\eta \otimes \mathbb{C})$, and $[\eta \otimes \mathbb{C}] \in \pi_{4n-1}(U(N))$ for some N large, actually, $\eta \otimes \mathbb{C} = i(\eta)$, where $i : SO(N) \to U(n)$. Thus, $c_{2n}(\eta \otimes \mathbb{C}) \in H^{4n}(S^{4n}, \pi_{4n-1}(St_{N,N-2n+1}(\mathbb{C}))) \cong \pi_{4n-1}(St_{N,N-2n+1})$ is the obstruction to extending an $N - 2n + 1$ complex framing of $(\eta \otimes \mathbb{C})$ to the southern hemisphere of S^{4n}, and since $[\eta \otimes \mathbb{C}]$ is the obstruction to extending the framing from the southern hemisphere to S^{4n}, it follows that $c_{2n}(\eta \otimes \mathbb{C}) = p_*(\eta \otimes \mathbb{C})$, where $p : U(N) \to U(N)/U(2n-1) \cong St_{N,N-2n+1}(\mathbb{C})$ is the projection.

So there is an exact sequence

$$\pi_{4n-1}(U(N)) \xrightarrow{p_*} \pi_{4n-1}(St_{N,N-2n+1}(\mathbb{C})) \xrightarrow{\partial} \pi_{4n-2}(U(2k-1)) \to \pi_{4n-2}(U(N))$$

From theorem (1.2.1)

$$\mathbb{Z} \xrightarrow{p_*} \mathbb{Z} \to \mathbb{Z}_{(2n-1)!} \to 0$$

hence p_* is the multiplication by $(2n-1)!$.

On the other hand,

$$\pi_{4n-1}(SO(N)) \xrightarrow{i_*} \pi_{4n-1}(U(N)) \xrightarrow{p_*} \pi_{4n-1}(St_{N,N-2n+1})$$

this composition maps $[\eta]$ to $\pm c_{2n}(\eta \otimes \mathbb{C}) = \pm p_n([\eta])$, so we have proved that $p_n([\eta]) = \pm i_*(2n-1)![\eta]$.

It remains to show that i_* is the multiplication by a_n. We have an exact sequence (for the stable groups)

$$\pi_{4n}(U/SO) \to \pi_{4n-1}(SO) \xrightarrow{i_*} \pi_{4n-1}(U) \to \pi_{4n-1}(U/SO)$$

Since by (1.2.1) $\pi_{4n}(U/SO) \cong \pi_{4n-2}(SO) \cong 0$, $\pi_{4n-1}(SO) \cong \pi_{4n-1}(U) \cong \mathbb{Z}$ and $\pi_{4n-1}(U/SO) \cong \pi_{4n-3}(SO)$ and $\pi_{4n-1}(U/SO) \cong \pi_{4n-3}(SO)$. The result follows from the fact that

$$\pi_{4n-3}(SO) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases}$$

□

We get the following results immediately from this theorem.

Corollary 4.3.13.

- $\sigma_{M,f}$ is independent of f.
- $p_n(M) = \pm a_n(2n-1)!\sigma_{M,f}$.
- $\sigma(M) = \frac{\pm a_n 2^{2n-1}(2^{2n-1}-1)B_n\sigma_{M,f}}{n}$.
- M is stably parallelizable if $\sigma(M) = 0$.

□

Definition 4.3.14. Let m and l be non negative integers, and define $J = J_{n,l} : \pi_m(SO(l)) \to \pi_{m+l}(S^l)$ as follows. If $[\alpha] \in \pi_m(SO(l))$, it can be represented by a family of isometries $\alpha_x \in SO(l)$ for each $x \in S^m$. View $S^{l+m} = \partial(D^{m+1} \times D^l) = S^m \times D^l + D^{m+1} \times S^{l-1}$ and $S^l = D^l/\partial D^l$. Let $J[\alpha] = \alpha_x(y)$ for $(x, y) \in S^m \times D^l$ and $J(\alpha)(D^{m+1} \times S^{l-1}) = \partial D^l$.

Proposition 4.3.15. J is a homomorphism.

Proof. Its clear that $\alpha \sim \beta$ implies $J(\alpha) = J(\beta)$. Now, we can view $J(\alpha)$ as a map $I^{m+l} \to S^l = D^l/\partial D^l$ which on $S^m \times D^l$ is given $(x, y) \mapsto \alpha_x(y)$ and sends the complement of $S^m \times D^l$ to ∂D^l. A similar view of β, the sum $J(\alpha) + J(\beta)$ is putting these two maps on either side of a hyperplane. Assume that α_x is the identity map for x in the northern hemisphere of S^m and β_x is the identity for x in the southern hemisphere of S^m. So there is a homotopy from $J(\alpha) + J(\beta)$ to $J(\alpha + \beta)$ by moving the two $S^m \times D^l$ together until they coincide; such as the figure 4.1 illustrates.

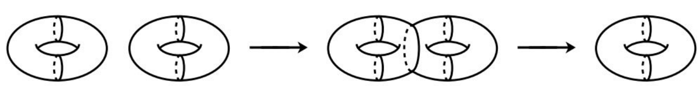


Figure 4.1: Homotopy between $J(\alpha) + J(\beta)$ and $J(\alpha + \beta)$

Taking $l \to \infty$, by (1.2.1), we have a well defined homomorphism

$$J_n : \pi_n(SO) \to \Pi_n.$$

It is commonly known as the *J-homomorphism.* □

Theorem 4.3.16. Let $\alpha \in \pi_{m-1}(SO)$, There exists an almost framed closed manifold (M^m, f) with $\sigma_{M,f} = \alpha$ if and only if $J(\alpha) = 0$

Proof. Suppose that (M, f) is an almost framed closed m-manifold, we may assume that f is a framing of $M - \text{int}D^m$. Choose an embedding of M in $\mathbb{R}^N$ so that D^m is the northern hemisphere of $S^m \subseteq \mathbb{R}^N$. Let f_0 be the usual normal framing of $D^m \subseteq \mathbb{R}^N$ and f_α the framing obtained by mapping $x \in S^{m-1}$ to

$\alpha(x)f_0(x)$.

The Thom–Pontryagin construction applied to the framed manifold (S^{m-1}, f_α) gives $\pm J(\alpha)$. Since $\alpha = \sigma_{M,f}$, and $f = f|_{S^{m-1}}$. we get that $(S^{m-1}, f_\alpha) = \partial(M - \mathrm{int}(D^m), f)$, and therefore (S^{m-1}, f_α) is null cobordant. That is, $J(\alpha) = 0$.

Conversely, suppose that $J(\alpha) = 0$, and set $S^{m-1} \subseteq D^m$. Let f_0 be the standard framing of $D^m \subseteq S^n$ for N large. Since $J(\alpha) = 0$, there is a framed manifold (N^m, f) such that $\partial(N, , f) = (S^{m-1}, f\alpha)$. Define $M = N + D^m$ pasting them throughout the boundary S^{m-1}. Then (M, f) is an almost framed closed manifold with $\sigma_{M,f} = \alpha$. □

Let j_n be the order of the image of the homomorphism $\mathbb{Z} \cong \pi_{4n-1}(SO) \xrightarrow{J} \Pi_{4n-1}$. We get the following results.

Corollary 4.3.17.

- The possible values for $\sigma_{M,f}$ are the multiples of j_n.
- The possible values for $\sigma(M)$ are the multiples of

$$\frac{a_n 2^{2n-1}(2^{2n-1} - 1)B_n j_n}{n}.$$

□

To finish the computation of the group bP^{4n} we use a difficult result, see [Ad].

Theorem 4.3.18 (Adams Theorem)**.** Let $J : \pi_m(SO) \to \pi_m(S)$. If $m \neq 4n$, J is injective, moreover j_m is equal to the denominator of $B_m/4m$. □

Corollary 4.3.19. If M is an almost framed closed manifold of dimension $\neq 4n$, then the almost framing of M extends to a complete framing. □

Corollary 4.3.20. $bP_{4n} = \mathbb{Z}_{t_n}$ where $t_n = a_n 2^{2n-2}(2^{2n-1} - 1)\cdot$ numerator of $B_n/4n$. □

From corollary 3.3.11,

Corollary 4.3.21. If $g_n \in bP_{4n}$ denotes the generator, then the manifold $\Sigma(3, 6k-1, 2, \ldots, 2)$ represents the element $(-1)^n k g_n$. □

The group bP_{2n}, n odd

Some of the theory needed to compute this group is developed in Appendix B.

In order to compute bP_{2n} for n odd and $n \neq 3, 7$ (Recall that $bP_6 = bP_{14} = 0$ from theorem (4.3.3)). We want to define a map

$$b_n : \mathbb{Z}_2 \to bP_{2n}$$

by $b_n(t) = \partial M$, where M is any compact framed $(n-1)$-connected $2n$-manifold with ∂M a homotopy sphere and $c(M) = t$.

Theorem 4.3.22.

1. Let M_1, M_2 manifolds as above, if $c(M_1) = c(M_2)$ then ∂M_1 is h-cobordant to ∂M_2.

2. For each $t \in \mathbb{Z}_2$ there is a framed manifold M^{2k} such that ∂M is a homotopy sphere and $c(M) = t$.

In particular this theorem implies that b_k is well-defined and is surjective.

Proof. 1. Let f_1 and f_2 be framings for M_1 and M_2 respectively. Let $(W, g) = (M_1, f_1)\#(M_2, f_2)$ be the framed boundary connected sum. $\partial W = \partial M_1 \#(-\partial M_2)$ and $c(W) = c(M_1) + c(-(M_2)) = 0$. By theorem (B.26), W is χ-equivalent to a contractible manifold. Thus $\partial M_1 \# \partial(-M_2)$ bounds a contractible manifold.

2. If $t = 0$, set $M^{2k} = D^{2k}$. If $t = 1$, consider the exotic sphere constructed in (3.3.15).

□

This shows that the group bP_{2n} is zero or isomorphic to a cyclic group of order 2, In particular, given a framed $(n-1)$-connected $2n$-manifold M with boundary the standard $(2n-1)$-sphere, an almost framed closed $2n$-manifold N can be obtained from M by attaching a disk to the boundary such that they both have the same Arf-Kervaire invariant; and viceversa. Since this is an invariant of framed cobordism, it follows that:

$bP_{2n} = 0$ *if and only if* there exists a closed framed $2n$-manofold M satisfying $c(M) = 1$.

Remark. Let M be a closed framed $2n$-manifold. *The Kervaire invariant problem* is the problem of determine for which values of n (odd) the Arf-Kervaire invariant $c(M)$ is non-trivial. By theorem (4.3.3), $c(M)$ is non-trivial when $n = 1, 3, 7$. Browder [Bw1] showed that $c(M)$ is non-trivial if and only if $n \neq 2^l - 1$. Mahowald and Tangora [MT] showed that $c(M)$ is non-trivial when $n = 15$. Barrat, Mahowald and Jones [BJM] showed that $c(M)$ is non-trivial when $n = 31$. Finally, Hill, Hopkins and Ravenel [HHR] showed that $c(M)$ is trivial for all $n \neq 1, 3, 7, 15, 31, 63$. Thus only the case $n = 63$ remains open.

Summarizing,

Theorem 4.3.23.

$$bP_{2n} = \begin{cases} 0 & \text{if } n = 1, 3, 7, 15, 31, \text{(and possibly) } 63 \\ \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

4.4 The group Θ_n/bP_{n+1}

Let Σ^n be a homotopy sphere embedded in $\mathbb{R}^{n+k}$ and f a framing of its normal bundle. The Thom–Pontryagin construction applied to (Σ, f) is an element $T(\Sigma, f) \in \pi_{n+k}(S^k)$, which is an invariant of the normal framed cobordism class of (Σ, f). Recall that $T((\Sigma, f)\#(\Sigma, f')) = T(\Sigma, f) + T(\Sigma', f')$.

Lemma 4.4.1. Let $f : \Sigma^n \to SO(k)$ and let $\alpha = [f] \in \pi_n(SO(k))$. If f' is the modification of f through α (compare the proof of (4.3.16)).

$$T(\Sigma, F') = T(\Sigma, f) \pm J(\alpha).$$

Proof. Since $T(S^n, f_\alpha) = \pm J(\alpha)$, where f_α is the modification of f_0 of S^n through α. Thus

$$(\Sigma, f') = (\Sigma, f')\#(S^n, f_0) = (\Sigma, f)\#(S^n, f_\alpha).$$

Applying T to both sides of the equation we get the result. □

Corollary 4.4.2. $T(\Sigma) = \{T(\Sigma, f), f \text{ is a framing of } \Sigma^n \subseteq \mathbb{R}^{n+k}\}$ is a coset of $J(\pi_n(SO(k)))$ in $\pi_{n+k}(S^k)$.

Define $T : \Theta_n \to \text{Coker}(J_n)$, where $J_n : \pi_n(SO) \to \pi_n(S)$ is the J-homomorphism.

Theorem 4.4.3. $bP_{n+1} = ker(T)$.

Proof. $\Sigma \in bP_{n+1}$ if and only if Σ bounds a parallelizable manifold. $T(\Sigma) = 0$ if and only if there exists a normal framing f of Σ such that (Σ, f) bounds a normally framed manifold. □

We have an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \xrightarrow{T} Coker(J_n).$$

Corollary 4.4.4. Θ_n is a finite group.

Consider the following group.

$$G_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ \mathbb{Z} & \text{if } n = 0 \bmod 4 \\ \mathbb{Z}_2 & \text{if } n = 2 \bmod 4 \end{cases}$$

And the homomorphism $b_n : G_{n+1} \to \Theta_n$

$$b_n(t) = \begin{cases} 0 & \text{if } n+1 \text{ odd} \\ [\partial M^{n+1}]\text{where } \sigma(M) = 8t & \text{if } n+1 = 0 \bmod 4 \\ [\partial N^{n+1}]\text{where } c(N) = t & \text{if } n+1 = 2 \bmod 4 \end{cases}$$

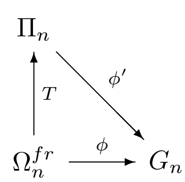
With $\partial M, \partial N$ homotopy spheres. From the computations made on section 4.3, $Im(b_n) = bP_{n+1}$.

Define now a map $\phi_n : \Omega_n^{fr} \to G_n$ as follow. For a class $[(M, \varphi)] \in \Omega_n^{fr}$, set

$$\phi(M, \varphi) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \sigma(M) & \text{if } n = 0 \bmod 4 \\ c(M, \varphi) & \text{if } n = 2 \bmod 4 \end{cases}$$

ϕ is well defined since σ and c are invariants under cobordism. And $\phi(M, \varphi) = 0$ if and only if (M, φ) is framed cobordant to a homotopy sphere.

Let $\phi' = \phi \circ T^{-1}$. so there is a commutative diagram



Note that $\phi'(Im(J_n)) = 0$, so ϕ' induces a map $\phi'' : Coker(J_n) \to G_n$.
In other words, we have proved.

Theorem 4.4.5. The sequence

$$G_{n+1} \xrightarrow{b_n} \Theta_n \xrightarrow{T} Coker(J_n) \xrightarrow{\phi''} G_n$$

is exact. □

Corollary 4.4.6.

$$\Theta_n/bP_{n+1} \cong ker(\phi'').$$

Remark. If n is odd, $\phi'' = 0$ since $G_n = 0$. If $n = 0$ mod 4, we have shown that $\phi'' = 0$. If $n = 2$ mod 4 then $\phi'' = 0$ except in the cases $n = 6, 14, 30, 62$ and possibly $n = 126$, by theorem (4.3.23).

This summarizes in the following result.

Theorem 4.4.7. For $n \geq 4$, $n \neq 2$ mod 4, there is an exact sequence

$$0 \to bP_{n+1} \to \Theta_n \to \text{coker}(J_n) \to 0.$$

If $n = 2$ mod 4, the exact sequence is given by

$$0 = bP_{n+1} \to \Theta_n \to J_n \xrightarrow{h} \mathbb{Z}_2 \to bP_n.$$

Where h is nonzero for $n = 6, 14, 30, 62$ and possibly $n = 126$. □

Chapter 5

Miscellaneous

In this chapter is included the computation of $|\theta_n|$ in specific cases using the results given in chapter 4. So we need to compute the order of the groups $|bP_{n+1}|$ and $|\theta_n/bP_{n+1}|$; these groups require the order of the groups Π_n, $\pi_n(SO)$ and $Im(J_n)$ (refer to [Ad]) and the sequence given by the Bernoulli numbers.

Bernoulli Numbers

n	1	2	3	4	5	6	7
B_n	$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$

The J-homomorphism

The order of the groups Π_n was taken from [Rv]. Using (1.2.1) and (4.3.18) we have,

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\|\pi_n(SO)\|$	2	1	∞	1	1	1	∞	2	2	1	∞	1	1	1	∞
$\|\Pi_n\|$	2	2	24	1	1	2	240	4	8	6	504	1	3	4	960
$\|Im(J_n)\|$	2	1	24	1	1	1	240	2	2	1	504	1	1	1	480
$\|Coker(J_n)\|$	1	1	1	1	1	2	1	2	4	6	1	1	3	4	2

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\|\pi_n(SO)\|$	2	2	1	∞	1	1	1	∞	2	2	1	∞	1	1	1
$\|\Pi_n\|$	16	4	16	528	24	4	4	3.144.960	4	4	12	24	2	3	6
$\|Im(J_n)\|$	2	2	1	264	1	1	1	65.520	2	2	1	24	1	1	1
$\|Coker(J_n)\|$	2	8	16	2	24	4	4	48	2	2	12	1	2	3	6

Groups of Homotopy Spheres

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\lvert bP_{n+1}\rvert$	1	1	1	1	1	1	28	1	2	1	992	1	1	1	8128	1	2
$\lvert\Theta_n/bP_{n+1}\rvert$	1	1	1	1	1	1	1	2	4	6	1	1	3	2	2	2	8
$\lvert\Theta_n\rvert$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256	2	16

n	18	19	20	21	22	23	24	25	26	27	28	29	30
$\lvert bP_{n+1}\rvert$	1	$2^8(2^9-1)$	1	2	1	$691\cdot 2^{11}(2^{11}-1)$	1	2	1	$2^{12}(2^{13}-1)$	1	1	1
$\lvert\Theta_n/bP_{n+1}\rvert$	16	2	24	4	4	48	2	2	12	1	2	3	6
$\lvert\Theta_n\rvert$	16	$2^9(2^9-1)$	24	8	4	$2073\cdot 2^{15}(2^{11}-1)$	2	4	12	$2^{13}(2^{13}-1)$	2	3	6

Recall that the class of exotic spheres up to diffeomorphism in dimension 4 does no coincides with Θ_4, so the number of classes of 4-dimensional exotic spheres is still unknown.

Appendix A

Spherical Modifications and Framed Cobordism

Notice that the manifold $S^p \times S^q$ can be considered either as the boundary of $S^p \times D^{q+1}$ or the boundary of $D^{p+1} \times S^q$. Given any imbedding of $S^p \times D^{q+1}$ in manifold M of dimension $n = p + q + 1$, a new manifold M' can be constructed by removing the interior of $S^p \times D^{q+1}$ and replacing it by the interior of $D^{p+1} \times S^q$ pasting them by the common boundary $S^p \times S^q$.

Definition A.1. Let $\varphi : S^p \times D^{q+1} \to M$ a smooth, orientation preserving imbedding. Let $\chi(M, \varphi)$ denote the quotient manifold obtained from the disjoint sum

$$(M - \varphi(S^p \times D^{q+1})) + (D^{p+1} \times S^q)$$

by identifying $\varphi(u, tv)$ with (tu, v) for each $u \in S^p$, $v \in S^q$, $0 < t \leq 1$. If M' denotes any manifold which is diffeomorphic to $\chi(M, \varphi)$ (with orientation preserving diffeomorphism) we say that M' is obtained from M by the *spherical modification* $\chi(\varphi)$ of type $(p + 1, q + 1)$

The boundary of M is equal to the boundary of $\chi(M, \varphi)$. Setting the notation $D^0 = \{0\}$ and $S^{-1} = \emptyset$, a spherical modification of type $(0, n+1)$ over M is the manifold $M + S^n$. Furthermore, if $M' = \chi(M, \phi)$ is obtained by a spherical modification of type $(p+1, q+1)$ then M can by obtained from M' by a spherical modification of type $(q + 1, p + 1)$.

Definition A.2. Let M, N two compact and oriented manifolds without boundary of the same dimension. M is χ-equivalent to N if there exists a sequence $M_0, M_1, \ldots, M_k$ with $M \cong M_0$, $N \cong M_k$ and such that each M_{i+1} can be obtained from M_i by a spherical modification.

Theorem A.3. Two such manifolds are $\chi-$equivalent if and only if the belong to the same cobordism class.

Proof. See [M3]. □

From the Thom–Pontryagin theorem we have

Corollary A.4. The Stiefel-Whitney numbers, Pontryagin numbers and the signature of a compact manifold M are invariant under spherical modifications. □

Let M be a connected manifold of dimension n and $\varphi : S^p \times D^{n-p} \to M$ an imbedding. Denote by $\lambda \in \pi_p(M)$ the homotopy class of the map $\varphi|_{S^p\times 0}$.

Lemma A.5. Let M a connected manifold of dimension n, with $n \geq 2p+2$ and $M' = \chi(M, \varphi)$ a spherical modification of type $(p+1, n-p)$ of M. Then the homotopy groups $\pi_i(M')$ are isomorphic to $\pi_i(M)$ for $i < p$ and $\pi_p(M') \cong \pi_p(M)/\Lambda$, where Λ denotes a certain subgroup which contains λ.

Proof. Set $q = n - p - 1$. Note that $p < q$. Let X denote the space $M + (D^{p+1} \times D^{q+1})$ by identifying (u, y) with $\varphi(u, y)$ for $(u, y) \in S^p \times D^{q+1}$. The subset $W \cup (D^{p+1} \times 0)$ is a deformation retract of X. Observe that this last subset is obtained from W by attaching a $(p+1)-$cell using the map $u \mapsto \varphi(u, 0)$, So the inclusion map $\pi_i(M) \to \pi_i(X)$ is an isomorphism for $i < p$ and is onto for $i = p$. Moreover, the homotopy class λ of the attaching map lies in the kernel of this homomorphism.
A similar argument shows that $\pi_i(M') \to \pi_i(X)$ is an isomorphism for $i < q$, since $p < q$ we get the result combining with the above paragraph. □

Proposition A.6. Let ξ be an m-plane bundle over a CW-complex X of dimension $p < m$. Then ξ is a trivial bundle if and only if $\xi \oplus \epsilon^1$ is trivial.

Proof. An isomorphism $\xi \oplus \epsilon^1 \cong \epsilon^{m+1}$ gives rise to a bundle map f from ξ to the bundle $\gamma^m(\mathbb{R}^{m+1})$, since the base space X has dimension p less than the dimension of the base space of γ^m, it follows that f is null homotopic, and hence ξ is trivial. □

Lemma A.7. Let M be a n dimensional compact manifold which is stably parallelizable, and let $\lambda \in \pi_p(M)$ where $p \leq n/2$. Then there exists an imbedding $\varphi : S^p \times D^{n-q}$ which represents λ. Moreover, φ can be chosen so that the manifold $\chi(M, \varphi)$ will also be stably parallelizable.

Proof. Since $p < n/2$, λ can be represented by an imbedding $\varphi_0 : S^p \to M$. Let TS^p be the tangent bundle of S^p and ν^{q+1} denote its normal bundle in M. Then the Whitney sum $TS^p \oplus \nu^{q+1}$ can be identified with $\varphi_0^* TM$ which is trivial by hypothesis.

Since $TS^p \oplus \epsilon^1$ is trivial,

$$\epsilon^{p+1} \oplus \nu^{q+1} \cong \epsilon^1 \oplus TS^p \oplus \nu^{q+1} \cong \epsilon^1 \oplus \epsilon^n \oplus \epsilon^{n+1}.$$

To prove the fact that φ makes the manifold $\chi(M, \varphi)$ stably parallelizable, we need to extend a trivialization f of $TM \oplus \epsilon^1$ to a trivialization of $TW \oplus \epsilon^1$. Where

$$W = (M \times I) + (D^{k+1} \times D^{n-k})$$

identifying $S^k \times D^{n-k}$ with $\varphi(S^k \times D^{n-k}) \times 1$, $\partial W = \chi(M, \varphi) - M$.

The obstructions to this extension lie in the cohomology groups $H^{k+1}(W, M, \pi_k(SO(n+1)))$ which is non zero only in the case $k = p$. Thus the only obstruction to extend f is a cohomology class

$$\sigma(\varphi) \in H^{p+1}(W, M, \pi_p(SO(n+1))) \cong \pi_p(SO(n+1)).$$

So the spherical modification $\chi(M, \varphi)$ is stably parallelizable if and only if the obstruction $\sigma(\varphi) = 0$. Let $\alpha : S^p \to SO(q+1)$ be a differentiable map, and define

$$\varphi_\alpha : S^p \times D^{q+1} \to M$$

by $\varphi_\alpha(u, v) = \varphi(u, \alpha(u)(v))$. Thus φ_α is an embedding which represents the same homotopy class $\lambda \in \pi_p(M)$ as φ. We will show that,

- $\sigma(\varphi_\alpha) = \sigma(\varphi) + s_*(\alpha)$ where $s_* : \pi_p(SO(q+1)) \to \pi_p(SO(n+1))$ is induced by the inclusion.

Since $p \leq q$, the homomorphism $s_* : \pi_p(SO(q+1)) \to \pi_p(SO(n+1))$ is onto. Therefore, we can choose α so that

$$\sigma(\varphi_\alpha) = \sigma(\varphi) + s_*(\alpha)$$

is zero. □

Theorem A.8. Let M be a compact, connected, stably parallelizable of dimension $n \geq 2k$. Then M is χ-equivalent to a $(k-1)$-connected stably parallelizable manifold N.

Proof. By lemmas (A.12) and (A.14), choose an imbedding $\varphi : S^1 \times D^{n-1} \to M$ to obtain a stably parallelizable manifold $M' = \chi(M, \varphi)$ such that $\pi_1(M')$ is generated by fewer elements of $\pi_1(M)$. Thus after a finite number of steps, we can obtain a stably parallelizable manifold M'' which is 1−connected. Then, after a finite number of steps we obtain a stably parallelizable 2−connected manifold M'' . We continue on this way to obtain a $(k-1)-$ connected stably parallelizable manifold N. □

Definition A.9. A *framed manifold* (M, f) is an oriented stably parallelizable smooth manifold M together with a framing f of $TM \oplus \epsilon^1$. A *framed spherical modification* $\chi(\varphi, g)$ of (M, f) is a spherical modification $\chi(\varphi)$ of M together a framing g of $TW \oplus \epsilon^1$ satisfying $g|_M = f \oplus t^1$. Where

$$W = (M \times I) + (D^{k+1} \times D^{n-k})$$

identifying $S^k \times D^{n-k}$ with $\varphi(S^k \times D^{n-k}) \times 1$. Thus $\partial W = \chi(M, \varphi) + (-M)$.

Restricting g to $\partial W - M = M'$ we obtain a framed manifold (M', f'). So we can consider a corresponding definition of framed cobordism. Two closed oriented framed manifolds (M, f), (M', f') are *framed cobordant* if there is a compact framed manifold (W, g) such that $\partial W = M + (-M')$ and $g|_M = f$, $g|_{M'} = f'$. The set of framed cobordism classes of framed closed manifolds are an abelian group under the operation of connected sum.

If f is homotopic to f' then (M, f) is framed cobordant to (M, f').

Lemma A.10. Let $i : M \to \mathbb{R}^{n+k}$ be an embedding and let N be a large integer. If f is a trivialization of $TM \oplus \epsilon^N$, there exists a trivialization f' of $\nu(i)$ such that $f \oplus f' \sim t^{N+n+k}$ and this f' is unique up to homotopy. Conversely, if f' is a trivialization of $v(i)$, there exist an unique (up to homotopy) trivialization f of $TM \oplus \epsilon^N$ such that $f \oplus f' \sim t^{N+n+k}$.

Proof. Let ξ^k and η^l plane bundles over the manifold M with $l > n+1$, such that $\xi^k \oplus \eta^l \cong \epsilon^{k+l}$. It is sufficient to show that if f is a trivialization of ξ^k then there exists a trivialization f' of n^l, unique up to homotopy, such that $f \oplus f' \sim t^{k+l}$.

Now, f defines a map $\phi : M \to St_{k+l,k}$, since $St_{k+l,k}$ is $(l-1)$-connected (1.1.32) and $n < l$ implies that ϕ is null homotopic by considering the obstruction classes. Thus by the homotopy lifting property of $St_{k+l,1} \to St_{k+l,l}$, ϕ extends to a map $M \to St_{k+l,k+l}$. So f exists.

Suppose that g is another trivialization of η^l with $f \oplus g \sim t^{k+l}$, then f' and g differ by a map $\alpha : M \to SO(l)$. Thus $i \circ \alpha \sim 0$ where $i : SO(l) \to SO(k+l)$. But $i_* : \pi_i(SO(l)) \cong \pi_i(SO(k+l))$ for $i < l-1$. Since $n < l-1$, $i_*[\alpha] = 0$ implies $[\alpha] = 0$; that is, $g \sim f'$. □

Definition A.11. Suppose that (M_1, f_1), (M_2, f_2) are normally framed manifolds (f_k is a trivialization of the normal bundle $\nu(i_k)$, $i_k : M_i \to \mathbb{R}^N$ embedding). (M_1, f_1) and (M_2, f_2) are *normally framed cobordant* if there is a manifold W with $\partial W = M_1 + M_2$ and an trivialization g of the normal bundle $\nu(i)$ of an embedding $i : W \to \mathbb{R}^N \times I$ such that $\text{int } W \cap \partial(\mathbb{R}^N \times I) = \emptyset$ and $i|_{M_k} = i_k$ and $g|_{M_k} = f_k$.

Theorem A.12. The set of normally framed cobordism classes of closed normally framed manifolds of dimension n, forms a group $\Omega(fr, n)$ under the connected sum. Then

$$\Omega(fr, n) \cong \Omega_n^{fr} \cong \lim_{k\to\infty} \pi_{n+k}(S^k).$$

Proof. The first isomorphism follows from applying lemma (A.17), the second one is using the Thom–Pontryagin theorem to a (B, f)-manifolds where the (B, f)-structure is given by $B_k = \{*\}$. Since $TB_k \cong S^k$ we have the result. □

Theorem A.13. Let M be a compact, connected, framed manifold of dimension $2k+1$, $k > 1$ such that ∂M is either vacuous or a homology sphere. Then M is χ-equivalent to a k-connected manifold N.

Proof. By theorem (A.15) we have that M is χ-equivalent to a M' $(k-1)$-connected manifold. We use the fact that ∂M is either vacuous or a homology sphere to prove that M' can be χ-equivalent to a k-connected manifold. See [KM, Lemma 6.6].

□

Appendix B

The Arf–Kervaire Invariant

The Arf–Kervaire invariant is a cobordism invariant of framed $(4k+2)$-manifolds, taking values in $\mathbb{Z}_2$. This theory was initially studied by Kervaire in [K].

The intersection product $H_k(M,\mathbb{Z}) \otimes H_k(M,\mathbb{Z}) \to \mathbb{Z}$ is skew symmetric and has determinant ± 1. Choose a basis of $H_k(M)$, namely $\{\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_k\}$ such that the intersection matrix is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

By the Hurewicz theorem, the elements α_i can be represented by disjoint embedded spheres. Let $f : S^k \to M$ an embedding representing an element $\alpha \in H_k(M)$. Thus

$$f^*(TM) \cong TS^k \oplus \nu(f).$$

Let g be a framing of $\epsilon^n \oplus TM$, it gives a framing f^*g of $\epsilon^n \oplus f^*(TM)$. If f_0 denotes the usual framing of $\epsilon^{n-1} \oplus TD^{k+1}$, $f_0|_{S^k}$ gives a framing of $(\epsilon^{n-1} \oplus TD^{k+1})|_{S^k} = \epsilon^n \oplus TS^k$.

The isomorphism,

$$\epsilon^n \oplus f^*TM \cong \epsilon^n \oplus TS^k \oplus \nu(f)$$

implies that the framing f^*g is a trivialization of the bundle $\epsilon^n \oplus f^*TM \cong \epsilon^n \oplus TS^k \oplus \nu(f)$. Thus the framing $f_0|_{S^k}$ assigns to each point in S^k an element of $St_{2k+n,k+n}$.

So, we can define an element

$$\phi(f) \in \pi_k(St_{2k+n,k+n}) \cong \mathbb{Z}_2 \ k \text{ odd}$$

which depends on M, g, f. Actually, if two embeddings f_1, f_2 represent the same element α, then $\nu(f_1) \cong \nu(f_2)$ and therefore we can define an element $\phi(\alpha) \in \mathbb{Z}_2$ independently of the choice of embedding.

Theorem B.1. 1. For $k \neq 3, 7$, $\phi(\alpha) = 0$ if and only if ν is trivial.

2. For $k = 3, 7$, $\nu(f)$ is trivial and $\phi(\alpha) = 0$ if and only if the spherical modification on M through $f : S^k \times D^k \to M$ can be framed.

Proof. 1. Consider the exact sequence in homotopy associated to the bundle $SO(k) \to SO(2k+n) \to St_{2k+n,k+n}$.

$$\cdots \pi_k(SO(k)) \xrightarrow{i_*} \pi_k(SO(2k+n)) \xrightarrow{p_*} \pi_k(St_{2k+n,k+n}) \xrightarrow{\partial} \pi_{k-1}(SO(k)) \xrightarrow{i_*} \cdots$$

From the definition, $\partial\phi(\alpha) = [\nu(f)] \in \pi_{k-1}(SO(k))$. For $k \neq 3, 7$, i_* is surjective so p_* is 0 and ∂ is injective. This implies

$$\phi(\alpha) = 0 \text{ if and only if } \partial\phi(\alpha) = [\nu(f)] = 0.$$

□

Define $\phi_2 : H_k(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ as the map

$$H_k(M, \mathbb{Z}_2) \to H_k(M, \mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{\phi \otimes id} \mathbb{Z}_2.$$

Definition B.2. Let V be a finite dimensional vector space over $\mathbb{Z}_2$ and $\langle , \rangle$ a symmetric bilinear form on V. A *quadratic function* is a function $\psi : V \to \mathbb{Z}_2$ such that

$$\psi(\alpha + \beta) = \psi(\alpha) + \psi(\beta) + \langle \alpha, \beta \rangle.$$

ϕ is called nonsingular if $\langle , \rangle$ is nonsingular. If $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ is a basis satisfying $\langle \alpha_i, \alpha_j \rangle = 0$ and $\langle \beta_i, \beta_j \rangle = \delta_{ij}$, the *Arf invariant* of $(\psi, \langle , \rangle)$ is defined by

$$A(\psi, \langle , \rangle) = \sum_i \psi(\alpha_i)\psi(\beta_i).$$

The definition is independent of the choice of this basis.

Proposition B.3. Let M, ϕ be as above. For $\alpha, \beta \in H_k(M, \mathbb{Z})$

$$\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta) + \langle \alpha, \beta \rangle \text{ mod } 2.$$

□

Corollary B.4. $\phi_2 : H_k(M, \mathbb{Z}_2) \to \mathbb{Z}_2$ is a nonsingular quadratic function associated to the intersection pairing.

Definition B.5. Let (M^{2k}, f), k odd, be a compact framed $(k-1)$-connected manifold with $H_k(M, \mathbb{Z})$ free abelian. The *Kervaire-Arf* invariant $c(M, f)$ is defined as

$$A(\phi_2, \langle , \rangle (\text{ mod } 2)) \in \mathbb{Z}_2.$$

For $c \neq 3, 7$, $c(M, f)$ does not depend on f.

Theorem B.6. Let (M^{2k}, f), k odd, be a compact framed $(k-1)$ connected manifold with ∂M a homotopy sphere. Then (M, f) is χ-equivalent to a contractible manifold if an only if $c(M, f) = 0$. □

Corollary B.7. Let Σ a homotopy sphere which bounds a stably parallelizable $2k$-manifold M with k odd. Then $c(M, f) = 0$ if and only if Σ is diffeomorphic to S^{2k-1}. □.

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