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Research Statement

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I perceive mathematics as a universal language, unraveling its significance as a mean to communicate with the universe and uncover its mysteries throughout the abstraction of ideas. I am mainly focused on studying the intricate symmetries of nature, so we can discover new properties by studying rotations, reflections or permutations of objects in a general set up. This motivation has driven me to get involved with the fields of algebra, topology and combinatorics. More specifically, I am interested on studying the equivariant cohomology of finite group actions on polyhedral products. My methods integrate and link knowledge from algebraic topology, group cohomology, transformation groups and toric topology among other areas in mathematics.

Background

Euler's famous formula for convex regular polyhedra (# Vert - # Edg + # Fac = 2) opened a gateway to major concepts in topology, namely, homology, Betti numbers and Euler´s characteristic. These topological and algebraic tools allow to classify closed orientable surfaces by genus for example. On the other hand, these topological invariants do not completely classify the symmetries (a.k.a group actions) of the space. For example, how can we tell the difference between the reflection, antipodal and π-radians rotation on the sphere S^2. If the action happens to be free, a good candidate is the quotient of the space by the action; however, this candidate does not fully capture the nature of the space and the group action.

In the 1960, Borel proposed an equivariant extension of the singular cohomology as follows [3]: If G is a topological group acting on a space X, define the G-equivariant cohomology of X as

$$H^*_G(X;R) := H^*(EG \times_G X;R)$$

where EG is the total space of the universal G-bundles. This algebraic structure is much richer than the ordinary cohomology ring of X starting from the fact that it is a module over the ring $H^*(BG;R)$. It turns out that freeness and torsion of this module is related to the freeness of the action of G on X. In the case that the G-equivariant cohomology of X is free over the cohomology of the classifying space BG, **we say that X is G-equivariantly formal over R.**

My research focuses on more intrinsic relations between the topology of the space, the nature of the action and the algebraic properties of the equivariant cohomology. I particularly study torus and finite group actions, restriction and transfer of the equivariant cohomology and canonical actions on combinatorial simplicial complexes. My research directions contain ideas that engage discussions and projects with mathematical enthusiasts from any level (from undergraduate students to colleagues in the same research area) in the following categories.

Reduction of the action to subgroups

Let G be a compact connected Lie group acting smoothly on a manifold M. The G-equivariant cohomology of M (with rational coefficients) is determined by the restriction of the action to a maximal torus T as they are related by the formula [6]

$$H_T^*(X) \cong H_G^*(X) \otimes_{H^*(BG)} H^*(BT).$$

In particular, this isomoprhims implies that X is G-equivariantly formal if and only if it is T-equivariantly formal. This reduction formula leads to the following questions:

- What are the algebraic properties of the groups that reduce equivariant formality to a suitable subgroup?
- Are there other geometric or algebraic properties of the space and the action that can be reduced to a subgroup?
- Are there more examples of pair of groups that allow such type of reduction?

These questions have allowed development of newly introduced concepts of free extension pairs, weak extension pairs and flat extension pairs. They study pairs (G, K) of a group G and a subgroup K such that $H^*(BK)$ is a free (or flat) module over $H^*(BG)$. I established a relation of these pairs and equivariant cohomology as summarized in the next result [4]

Theorem 1 *Let (G, K) be a free extension pair and X be a G space such that G acts trivially on the cohomology of X. Then there is an isomorphism*

$$H_K^*(X) \cong H_G^*(X) \otimes_{H^*(BG)} H^*(BK)$$

Moreover, X is G-equivariantly formal if and only if it is K-equivariantly formal.

A similar result also holds for weak and flat extension pairs. Since (G, T) is a free extension pair when G is a compact connected lie group and T is a maximal torus, Theorem 1 generalizes this situation to a further extend. Other examples of free and flat extension pairs include Torus, elementary abelian p-groups and cyclic groups.

Finite group actions

Let G be a cyclic group of order n and $\Bbbk$ be a field of characteristic $p \mid n$. Unlike torus actions where the cohomology ring of the classifying space is a polynomial ring, for the cyclic group case the torsion of the cohomology of BG does not allow the same methods to directly work. A workaround is to restrict to the polynomial subring $R \subseteq H^*(BG)$ and study the freeness of the equivariant cohomology over this subring, or almost-equivariantly formal spaces. Study for p-tori have been developed in [1]. Questions that raised for such a type of group actions are the following:

- What properties of equivariant formality carry over almost-equivariantly formal?
- Is it possible to characterize algebraically almost-equivariantly formal spaces?
- What is the relation with free and flat extension pairs of cyclic groups?

The findings of my research on this direction can be summarized in the following result [5]:

Theorem 2 *Let X be a G-space and $K \subseteq G$ be the subgroup of order p. Then*

- *If X is G-equivariantly formal, then it is K-equivariantly formal.*
- *If the action can be extended to a circle action, then the converse of the previous statement is true.*
- *(G, K) is a weak and a flat extension pair.*
- *If X is G-almost-equivariantly formal, it might not be K-almost equivariantly formal.*
- *The converse of the previous statement is true if the representation of G on $H^*(X)$ has norm zero.*

Actions on polyhedral products

Polyhedral products have served as a generalization of toric manifolds, moment-angle complexes and spaces associated to simplicial complexes [2]. A nice feature of such type of spaces is that the combinatorial and algebraic properties of the underlying simplicial complex carry over plenty of information about the topology of the space and the cohomological structures associated at it. My main focus is to study the polyhedral product $X_n(a) = Z_{K_n}(D^{a+1}, S^a)$ where K_n is the boundary of an n-gon (so $n \geq 3$) and $a \geq 0$. When $a = 0$, the polyhedral product is homeomorphic to a surface of genus $1 + (n-4)2^{n-3}$ that can be computed using Euler´s polyhedral formula. There is a canonical action of the cyclic group of order n, G on $X_n(a)$ that arises from the intrinsic rotation on the underlying polygon. One of the main focuses that my research has done in this direction is to determine explicitly the action of G on the cohomology of $X_n(a)$, and in particular, for $a = 0$, as a consequence of the previous results we can tell the values of n and p so $X_n(0)$ is equivariantly formal and almost-equivariantly formal. A lot of the methods used require a heavy load of the underlying combinatorics of the cyclic action of G on the polygon. Namely, this action is related to 2-colored necklaces and binary Lyndon words. It is possible to obtain combinatorial relations for these equivalent problems from an equivariant cohomology perspective.

Current research and future directions

My current and future research initiatives can be organized in the following categories:

- **Free and Flat extension pairs:** Any free extension pair is a weak extension pair and a flat extension pair. The only example that I have so far of a non-free extension pair which is a weak and a flat extension pair is in the case of finite cyclic groups. I am exploring other examples or the relation between these two last notions. Are they the same equivalent? does one imply the other? There are algebraic answers to such a problem; however, from a group cohomology perspective the answer is unknown.

- **Equivariant cohomology for finite groups:** The equivariant cohomology for compact connected lie groups is mostly determine by a torus actions. For finite abelian groups, there are relations with the maximal elementary p-subgroups. There are canonical actions of non-abelian finite groups (symmetric, dihedral, quaternion groups to mention) as well as cyclic groups. A natural question is to study the canonical permutation actions on an n-fold product of a space. This leads to studying polyhedral products, symmetric products and cyclic products. In particular, the cyclic product of spheres, their orbit space and equivariant cohomology it is yet to be understood.

- **Combinatorial relations of the action on polyhedral products:** I have determined the G-module structure for the cohomology of the space $X_n(a)$ in terms of combinatorial formulas involving binary Lyndon words. It is also known that the equivariant cohomology for $a-0$ is the E_2-term of the associated Borel spectral sequence. I am working on a similar result for any $a > 0$ and so classify the values of n and p so $X_n(a)$ is equivariantly formal (or almost equivariantly formal). An important objective to achieve for such class of spaces is to describe combinatorially the G-cohomology of the space in a similar way that it is done for the torus action on the case $a = 1$ in terms of the Stanley-Reisner ring associated to the simplicial complex K_n.

References

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