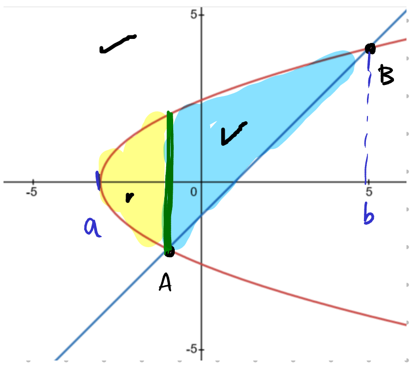
Concept Check: Area between curves

Example: Find the area of the region bounded by $y^2 = 2x + 6$ and $y = x - 1$.



1. Use x-integration to find the area of the region in the previous example.

- Identify the interval of integration [a,b].
- Identify the upper and lower curves. Are they always the same?
- How many integrals do you need to set up?
- Set up and evaluate the integrals. Compare your answer with the one obtained in class.

Solution: Find the intersection points between $y^2 = 2x+6$ and $y = x-1$

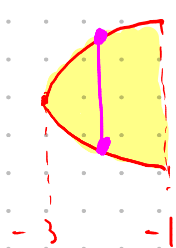
$A = (-1,-2)$ and $B = (5,4)$ ✓

To do x-integration, we integrate from $[a,b]$. The value of a is obtained when $y=0$ in $y^2 = 2x+6$. So $0 = 2x+6 \Rightarrow x = -3$.

The interval of integration is $[-3,5]$

The upper and lower curves are not always the same throughout $[-3,5]$, there are two types of vertical slices:

Type I: on the interval $[-3,-1]$:



Upper branch of $y^2 = 2x+6$
$y = \sqrt{2x+6}$

lower branch of $y^2 = 2x+6$
$y = -\sqrt{2x+6}$

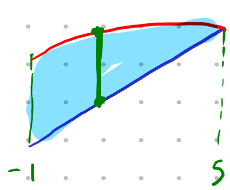
The area of this region is $A_1 = \int_{-3}^{-1} \left(\sqrt{2x+6} - (-\sqrt{2x+6})\right) dx = 2\int_{-3}^{-1} \sqrt{2x+6}\, dx$

use $u = 2x+6 \Rightarrow du = 2\,dx$

If $u = -3 \Rightarrow x = 0$
$u = -1 \Rightarrow x = 4$

$$A_1 = \int_0^4 u^{1/2}\, du = \left.\frac{u^{3/2}}{3/2}\right|_0^4 = \frac{2}{3}\cdot 8 = \frac{16}{3}$$

Type II on the interval $[-1,5]$



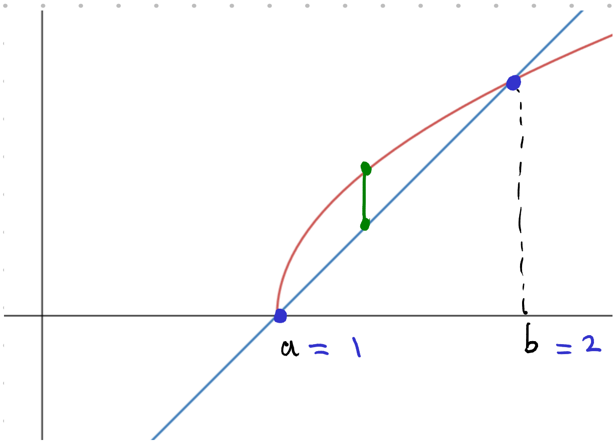
upper curve: $y = \sqrt{2x+6}$

lower curve: $y = x-1$

$$A_2 = \int_{-1}^{5} \sqrt{2x+6} - (x-1)\, dx = \left.\frac{u^{3/2}}{2}\right|_4^{16} - \left.\left(\frac{x^2}{2} - 1\right)\right|_{-1}^{5}$$

2. Find the area of the region enclosed by $y = \sqrt{x-1}$ and $x - y = 1$ using both methods of integration (x-integration and y-integration).

Using x-integration (vertical slicing)



Points of intersection: $y = \sqrt{x-1}$ and $y = x-1$

$$\sqrt{x-1} = x-1 \Rightarrow x-1 = (x-1)^2 = x^2 - 2x + 1$$

$$\Rightarrow x^2 - 3x - 2 = 0$$

$$(x-1)(x-2) = 0$$

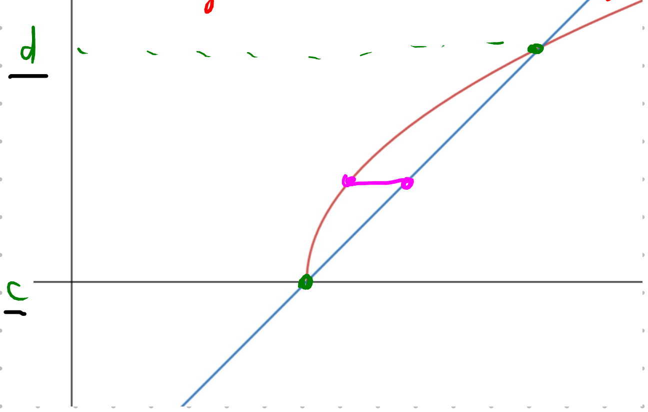
$$x = 1, \quad x = 2$$

One type of slice: upper curve: $y = \sqrt{x-1}$

lower ": $y = x-1$

$$A = \int_1^2 \sqrt{x-1} - (x-1)\, dx = \frac{(x-1)^{3/2}}{3/2} - \frac{x^2}{2} + x \Big|_1^2 = \left(\frac{2}{3} - 2 + 2\right) - \left(0 - \frac{1}{2} + 1\right) = \frac{1}{6}$$

y-integration (horizontal slicing)



$c = 0 \qquad d = 1$

Rightmost curve: $x = y+1$

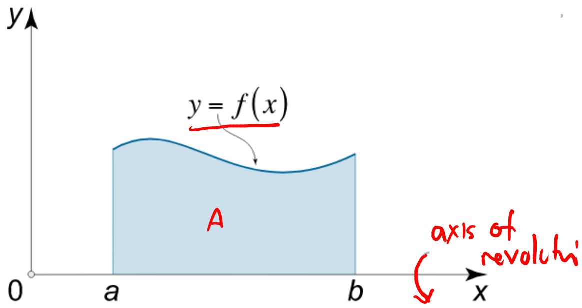
Leftmost curve: $x = y^2+1$

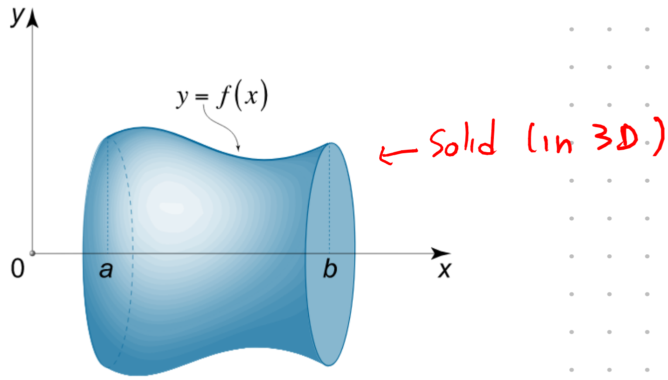
$$A = \int_0^1 y+1 - (y^2+1)\, dy = \int_0^1 y - y^2\, dy = \frac{y^2}{2} - \frac{y^3}{3} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

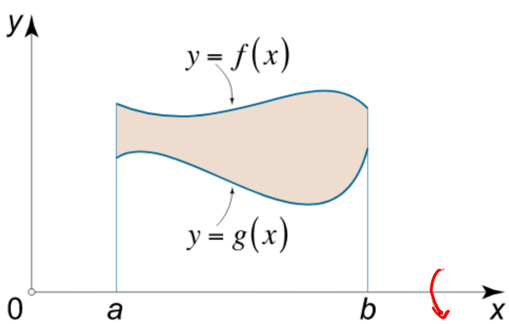
Solids of revolution and their volume

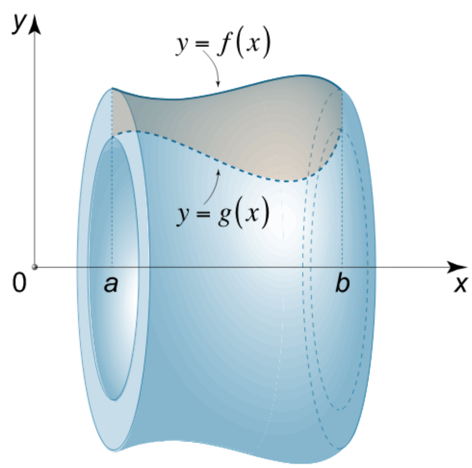
Revolving/rotating an area about the x-axis:



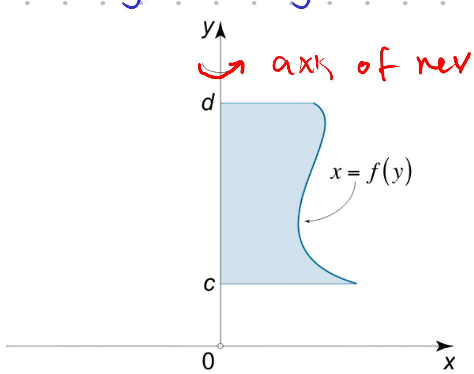


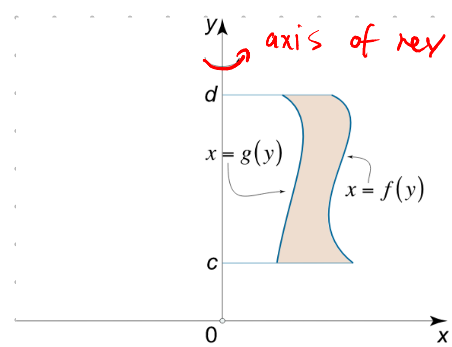
Revolving/rotating the area between two curves about the x-axis:

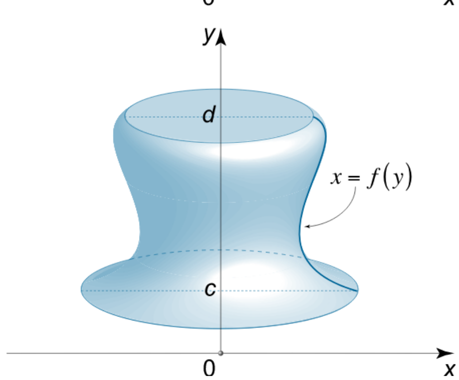


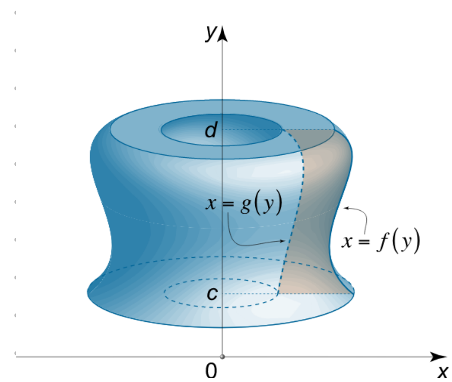


Revolving/rotating an area about the y-axis:



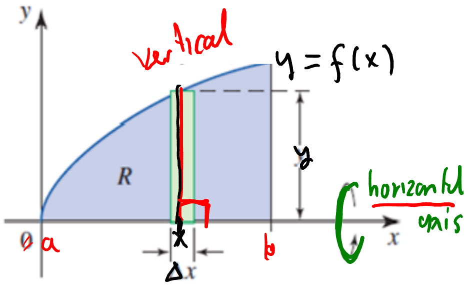


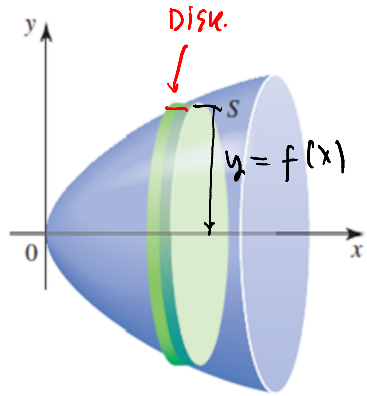




How to find the volume of a solid of revolution?

Method 1: The Disk/Washer method (Perpendicular slicing)



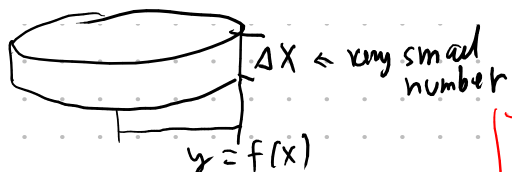


Find the volume of this disk and then integrate this volume from a to b

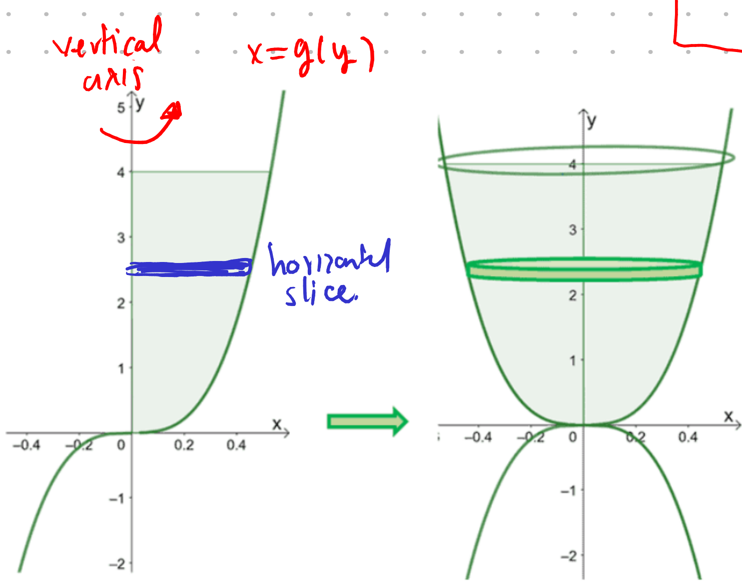
If $x \in [a, b]$

Volume of the slice $= \pi r^2 \cdot h$

$= \pi (f(x))^2 \cdot \Delta x$

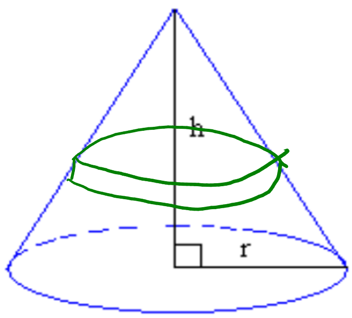


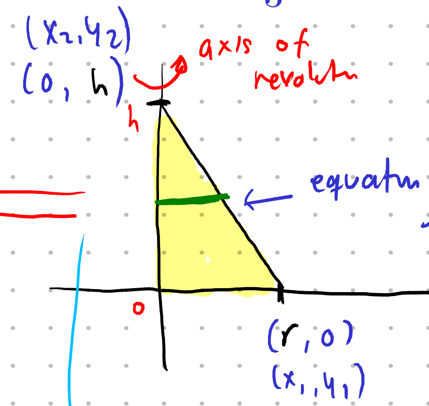
$$\boxed{V = \int_a^b \pi (f(x))^2 \, dx}$$



$$V = \int_c^d \pi \, g(y)^2 \, dy$$

Example: Show that the volume of a cone is $\frac{1}{3}\pi r^2 h$





slope $= m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{h-0}{0-r} = -\frac{h}{r}$

$y = mx + b$

$$\boxed{y = -\frac{h}{r}x + h}$$

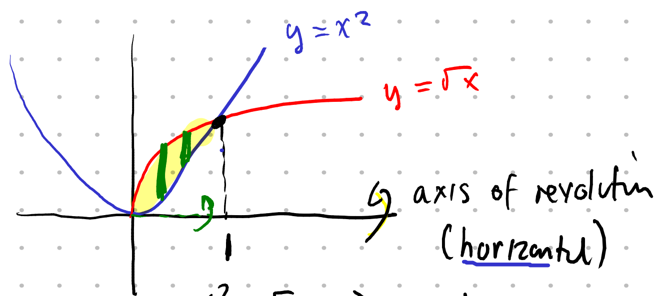
$x = -\frac{r}{h}(y - h)$

Horizontal slicing $\Rightarrow$ disks
+
vertical axis

$A = \int_0^h \pi\left(-\frac{r}{h}y + r\right)^2 dy = \pi\int_0^h \frac{r^2}{h^2}y^2 - \frac{2ry}{h} + r^2\, dy$ $\quad x = -\frac{r}{h}y + r \Rightarrow x = g(y)$

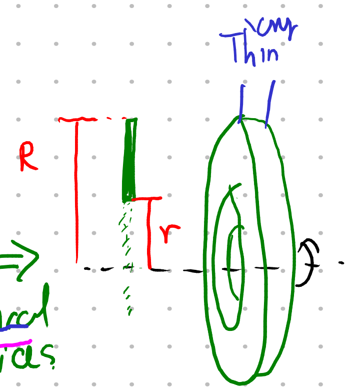
$= \pi\left(\frac{r^2}{h^2}\frac{y^3}{3} - \frac{r}{h}y^2 + r^2 y\right)\Big|_0^h = \cdots = \frac{\pi}{3}r^2 h$

Example: Find the volume of the solid obtained by revolving the area between $y = x^2$ and $y = \sqrt{x}$ about the x-axis.



$x^2 = \sqrt{x} \Rightarrow x = 1$

$\Rightarrow$ vertical slices



Disk with a hole
=
Washer

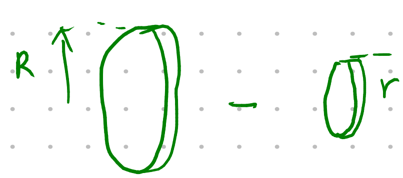
$A = \int_0^1 \pi\left(\text{upper curve}^2 - \text{lower curve}^2\right)dx$

$= \int_0^1 \pi\left(\sqrt{x}^2 - (x^2)^2\right)dx$

$= \pi\int_0^1 x - x^4\, dx = \pi\, \frac{x^2}{2} - \frac{x^5}{5}\Big|_0^1$

$= \pi\left(\frac{1}{2} - \frac{1}{5}\right)$

Volume of washer = Volume of the solid disk (R) − Volume of the hole (r)

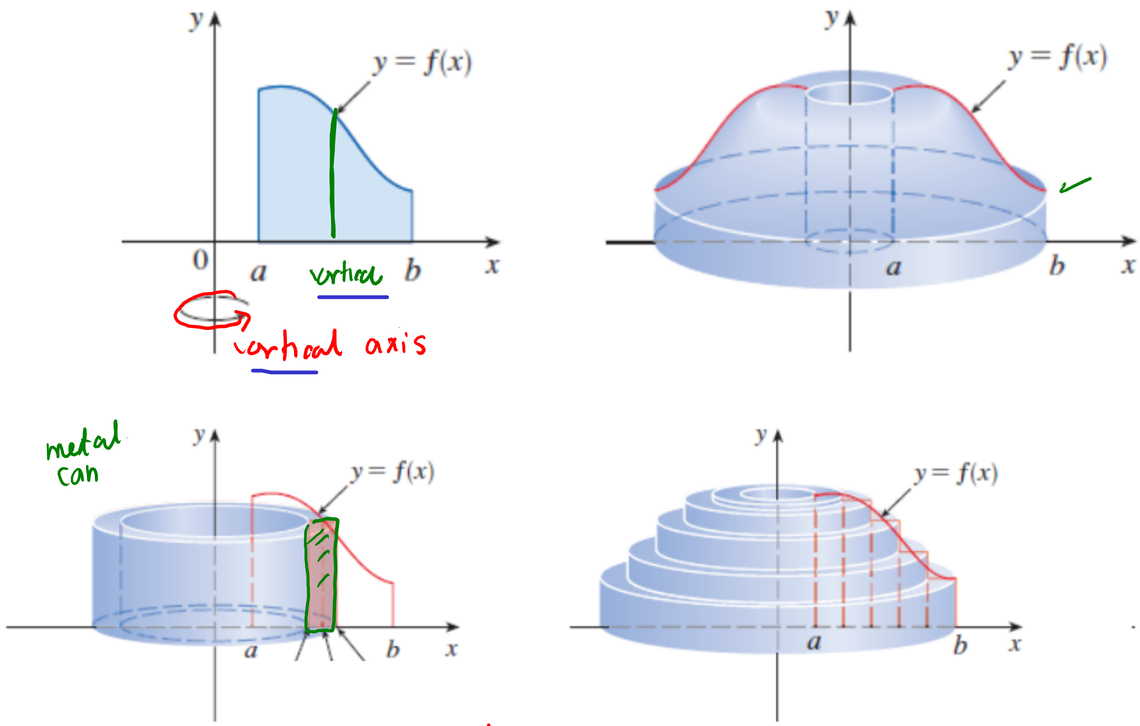


$V = \pi R^2 \Delta x - \pi r^2 \Delta x$

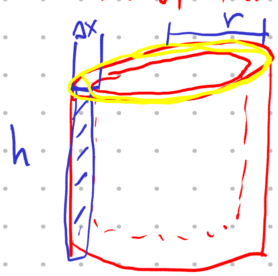
$$\boxed{= \pi(R^2 - r^2)\Delta x}$$

do not mistake by $\neq \pi(R-r)^2\Delta x$!!!!

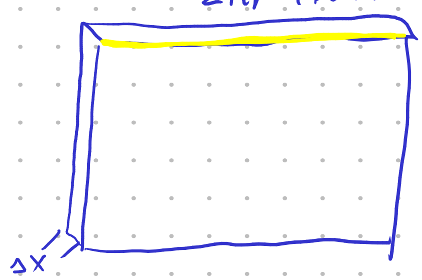
Method 2: The cylindrical shell method (Parallel slicing)



Volume of the cylindrical shell

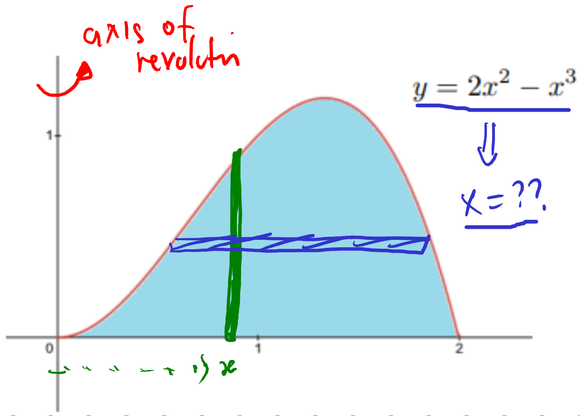


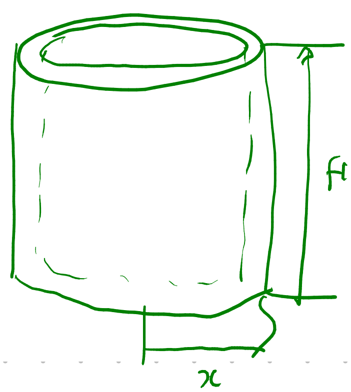
⟹ Tear it apart



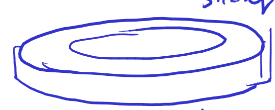
$$V = \underbrace{2\pi r}_{\text{base}}\ \underbrace{h}_{\text{height}}\ \underbrace{\Delta x}_{\text{width}}$$

Example: Find the volume of the solid obtained by revolving the area under $f(x) = 2x^2 - x^3$ in the first quadrant about the y-axis.





Problem with hor. slicing.



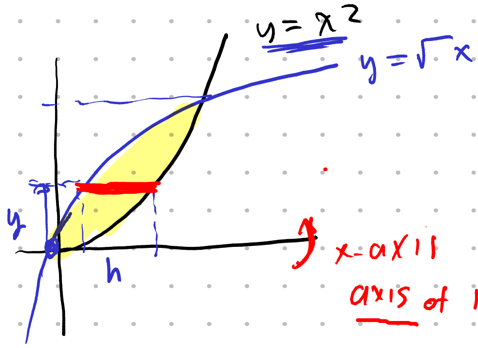
x can't be isolated

$$V = 2\pi\, x\, f(x)\, \Delta x$$

$$\boxed{V = \int_a^b 2\pi\, x\, f(x)\, dx} = \int_0^2 2\pi x\,(2x^2 - x^3)\,dx$$

$$2\pi \int_0^2 2x^3 - x^4\,dx = \dots$$

Example (revisited): Find the volume of the solid obtained by revolving the area between $y = x^2$ and $y = \sqrt{x}$ about the x-axis. Use the shell method



h = Rightmost curve − leftmost curve

$x = \sqrt{y}$ − $x = y^2$

$\sqrt{y} - y^2$

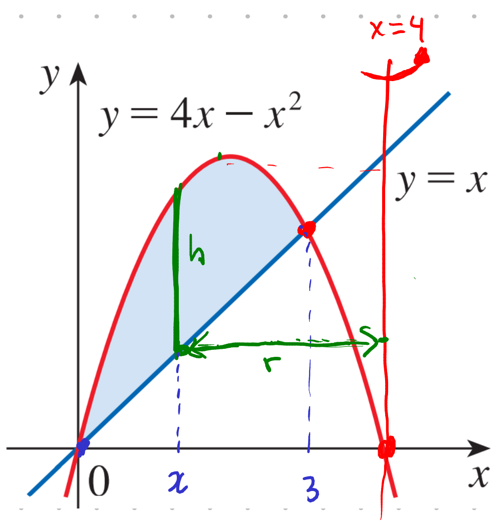
$$V = \int_0^1 2\pi\, y\left(\sqrt{y} - y^2\right) dy \checkmark = 2\pi \int_0^1 y^{3/2} - y^3\, dy$$

$$= 2\pi\left(\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right)$$

$$= 2\pi \cdot \left(\frac{2}{5} - \frac{1}{4}\right)$$

Revolving/rotating an area about different axes of revolution

Example: Consider the region enclosed by the following curves:



$y = 4x - x^2$ $\quad y = x$

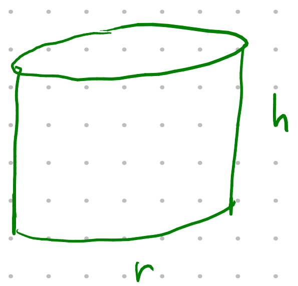
$4x - x^2 = x$

$3x - x^2 = 0$

$x = 0 \quad x = 3$

vertical axis + vertical slice = cylind. shell.

Find the volume of the solid obtained by revolving the area about the line x = 4



$h =$ upper curve − lower curve $= 4x - x^2 - x = 3x - x^2$

$r = 4 - x$

$V = 2\pi(4 - x)(3x - x^2)$

$$\boxed{V = \int_0^3 2\pi(4-x)(3x-x^2)\,dx} = 2\pi\int_0^3 4(3x-x^2) - x(3x-x^2)\,dx$$

$$= 2\pi\int_0^3 12x - 4x^2 - 3x^2 + x^3\,dx$$

Concept Check

Consider the region in the first quadrant enclosed by $y = x^2$ and $y = x^3$

Set up (not evaluate) an integral represents the volume of the solid obtained by revolving the area about the given axis of revolution. Use both methods of slicing on each volume

- The x-axis
- The y-axis
- The line x = -1
- The line y = 2