Solving First-Order Linear Differential Equations

Step 1. Write the DE in the general standard form

$$y' + p(x)y = q(x)$$

Step 2. Find the corresponding integrating factor

$$I(x) = e^{\int p(x)dx}$$

Step 3: The integrating factor yields the identity

$$(I(x)y)' = I(x)q(x)$$

Step 4: Integrate both sides of the above equation and solve for y to obtain

$$y = \frac{1}{I(x)}\left(\int I(x)q(x)dx + c\right)$$

Example: Solve: $(x^2 - 1)\dfrac{dy}{dx} + 2xy = x$

Note: Being careful with piecewise functions

Example 1.6.5 Solve the initial-value problem

$$y' - y = f(x), \qquad y(0) = 0,$$

where $f(x) = \begin{cases} 1, & \text{if } x < 1, \\ 2 - x, & \text{if } x \geq 1. \end{cases}$

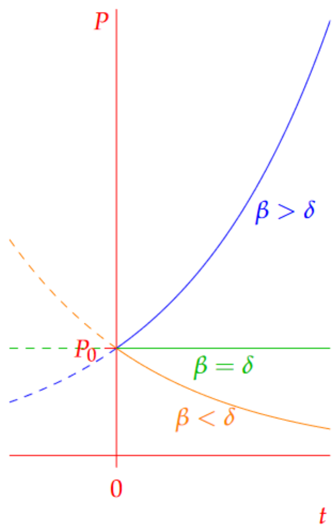
1.7 Applications of Differential Equations

Population Models

The rate of growth of the population is directly proportional to the population.

Let k be the growth constant rate of the population. Then $\frac{dP}{dt} = kP$

usually, k = b-d where b is the constant birth rate and d is the constant death rate.



General solution is

$$P(t) = P_0 e^{kt}$$

Example:

The population of a certain community was 10 000 in 1980. In 2020, it was found to have grown to 50 000. Form an exponential function to model the population of community P that changes through time t.

A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches 10,000. When will the owner's friends be allowed to fish?

Non-constant growth rate

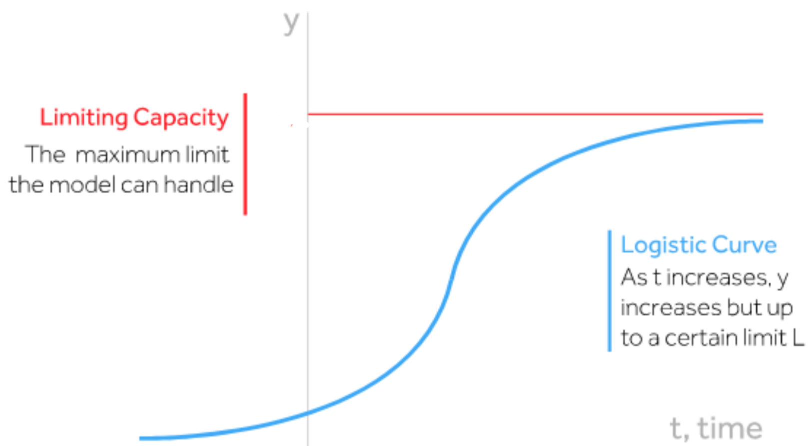
A population of 25 fish in a pond has a constant death rate $\delta = 0.5$ per fish, per year and a birth rate of $\beta(t) = 1.5 - 0.01P$ per fish, per year. How long it will take for the fish population to double? Can you foresee what is the outcome of this population?

Limited Resources: The logistic model

The enviroment where a population is developing is supposed to have a maximum population to sustain. It is called the Carrying capacity and denoted by M.

We introduce the growth inhibitor $\left(1-\frac{P}{M}\right)$ to get the model

$$\frac{dP}{dt}=kP\left(1-\frac{P}{M}\right)$$

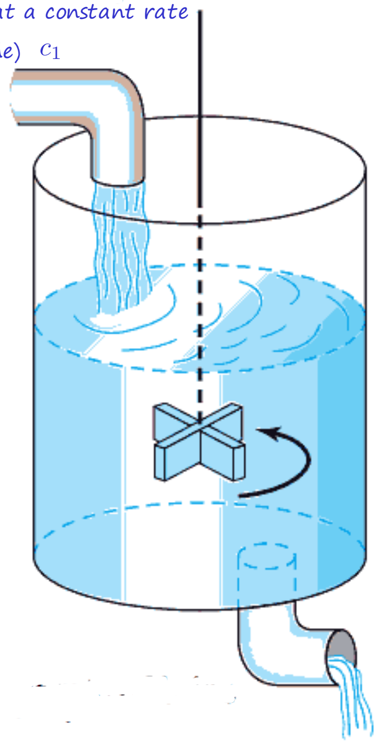


Example

A virus epidemic was formally announced by the city government. According to medical data, it took 5 days for the virus t
infect from 3 to 376 people. If the city has 100.000 residents, how many residents are predicted to be infected after 15
assuming no measures were taken yet by city officials?

Mixing Problems

Liquid mixed with a chemical flows in at a constant rate
and it has a concentration (mass/volume) c_1



There is initially a volume V_0

The volume of the liquid changes over time $V(t)$

The amount of chemical changes over time $A(t)$

Liquid flows out at a constant rate r_2

The concentration of the mixture in the tank changes over time.
What is the concentration (as a function of time)
of the mixture coming out? $c(t)$

How to answer this question?

- Use that $c(t) = \dfrac{A(t)}{V(t)}$ $\qquad \dfrac{dV}{dt} = \text{rate in - rate out} \qquad \dfrac{dA}{dt} = c_1 r_1 - c(t) r_2$

Example 1.7.1 A tank contains 8 L (liters) of water in which is dissolved 32 g (grams) of chemical. A solution containing 2 g/L of the chemical flows into the tank at a rate of 4 L/min, and the well-stirred mixture flows out at a rate of 2 L/min.

1. Determine the amount of chemical in the tank after 20 minutes.
2. What is the concentration of chemical in the tank at that time?

Example

A tank whose volume is 40 L initially contains 20 L of water. A solution containing 10 g/L of salt is pumped into the tank at a rate of 4 L/min, and the well-stirred mixture flows out at a rate of 2 L/min. How much salt is in the tank just before the solution overflows?

Learning Outcomes

- Identify the general form of a first order linear DE.
- Obtain the Integrating factor and reproduce the steps to solve a first order linear DE.
- Analyze the domain of a piecewise function when finding the general solution to a DE.
- Recognize the use of DE on Newton's law of cooling, Population models and Mixture problems.
- Design a DE model from a word problem using rates of change.