Assignment FinalExam due 06/22/2023 at 12:30pm EDT

Problem 1. 1. **(3 points)** `local/1/header-f.pg`

Final Exam - Discrete Mathematics : Summer 2023

Read all the instructions below before start working on the exam

- Copy down the Honor Pledge on your paper: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own* and **sign** besides it.
- You have 180 minutes (3hr) or until 12:00 pm (whichever comes first) to work on your exam.
- You can change your answers anytime before submitting your exam.
- To submit your answers, click on "Grade Test" at the bottom of this page. Clicking on that button **will prompt a pasword to finish the exam immediately**. Make sure you have entered all your answers before hitting "Grade Test".
- You have **ONLY one exam attempt**. Re-attempts won't be allowed under any circumstance.
- You must show all your work to receive full-credit. Copy down on your paper both final answers and justifications **for all problems on the exam**.
- Once the exam is over, scan all your solutions in a single PDF file (including the Honor Pledge) and post the solutions on Gradescope.
- When a prompt (user and passcode) is required when clicking on "Grade Test", enter the following information: User: *ExamDone* Passcode: *HappySummer*

  - I have read, understood and agreed to all the exam instructions

Correct Answers:

- `I ... instructions`

Problem 2. 2. **(6 points)** `local/11/100/logic/contrapositive-f.pg`

FINAL EXAM - PART A

Submit only final answers to gradescope

Consider the proposition

If it is not raining then either the sun is shining or it is not windy.

For each of the following propositions, determine if it is the contrapositive, converse, inverse or neither of the given proposition above.

1. *If it is raining then either the sun is shining or it is not windy*

   - **A**. inverse
   - **B**. converse
   - **C**. contrapositive
   - **D**. None of the above

2. *If today is raining, then neither the sun is shining nor it is not windy*

   - **A**. contrapositive
   - **B**. converse
   - **C**. inverse
   - **D**. None of the above

Solution: *(Instructor solution preview: show the student solution after due date.)*

Given the statement $p \to q$: If it is not raining then either the sun is shining or it is not windy

- The contrapositive is $\neg q \to \neg p$:If neither the sun is shining nor it is not windy, then it is raining.
- The converse is $q \to p$: If either the sun is shining or it is not windy, then it is not raining.
- The inverse is $\neg p \to \neg q$: If today is raining, then neither the sun is shining nor it is not windy.

The correct solutions are then:

- Part 1: D
- Part 2: C

Correct Answers:

- D
- C

Problem 3. 3. (9 points) local/11/100/logic/predicate-tf-f.pg

Submit justifications to gradescope to get full credit
For each of the following predicates, determine if they are True or False in the universe of **Integers**. Provide your explanations on Gradescope.

1. $\exists y\ \forall x\ x^y \equiv x\ (\mod 7)$

   - True
   - False

2. $\forall x\ \forall y\ (x|y \wedge y|x) \rightarrow x = y$

   - True
   - False

3. $\exists x\ \forall y\ x|y \wedge x^2 \neq x$

   - True
   - False

Solution: *(Instructor solution preview: show the student solution after due date.)*

- True: Take $y = 7$ by Fermat's little theorem
- False: Take $x = 1, y = -1$
- True: There exists $x = -1$

Correct Answers:

- True
- False
- True

Problem 4. 4. (8 points) local/11/100/logic/truth-tables-2.pg
Complete the following truth table to find the truth values of the formula

$$\alpha = ((p \vee \neg q) \wedge (\neg p \wedge r)) \rightarrow (q \vee \neg r)$$

by filling in the blanks with T (for True) or F (for False) as appropriate.

$\_\_p$	q	r	$(p \vee \neg q)$	$\neg p \wedge r$	$(p \vee \neg q) \wedge (\neg p \wedge r)$	$q \vee \neg r$	α
—T	T	T	___	___	___	___	___
—T	T	F	___	___	___	___	___
—T	F	T	___	___	___	___	___
—T	F	F	___	___	___	___	___
—F	T	T	___	___	___	___	___
—F	T	F	___	___	___	___	___
—F	F	T	___	___	___	___	___
—F	F	F	___	___	___	___	___

Based on the truth table, is α a tautology?

- **A**. It is a Tautology
- **B**. It is NOT a Tautology

Solution: *(Instructor solution preview: show the student solution after due date.)*

$\_\_p$	q	r	$(p \vee q)$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	α
—T	T	T	T	F	F	T	T
—T	T	F	T	F	F	T	T
—T	F	T	T	F	F	F	T
—T	F	F	T	F	F	T	T
—F	T	T	F	T	F	T	T
—F	T	F	F	F	F	T	T
—F	F	T	T	T	T	F	F
—F	F	F	T	F	F	T	T

Based on the truth table, The correct answer is B

Correct Answers:

- T
- F
- F
- T
- T
- T
- F
- F
- T
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- T
- F

- F
- F
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- F
- F
- T
- F
- F
- T
- T
- B

Problem 5. 5. (12 points) local/11/100/sets/proof-f.pg

Submit justifications to gradescope to get full credit

For each of the following statements determine if it is true or false. Provide a proof, counterexample or justification to earn full credit. Select the corresponding truth value here, **submit your justifications to gradescope and show all your steps**.

- 1. If the number n^3 is divisible by 3, then n is divisible by 3.
  - True
  - False
- 2. The number $\sqrt[3]{3}$ is irrational.
  - True
  - False
- 4. The University of Rochester has about 6000 students. There are at least 17 students that share the same birthday (month and day).
  - True
  - False

Solution: *(Instructor solution preview: show the student solution after due date.)*

- 1. True: By contrapositive, if n is not divisible by 3, there are two cases:
  $n \equiv 1 \mod 3 \rightarrow n^3 \equiv 1^3 = 1 \mod 3$.
  $n \equiv 2 \mod 3 \rightarrow n^3 \equiv 2^3 \equiv 2 \mod 3$
- 2. True: By contradiction. Assume $\sqrt[3]{3}$ is rational so write it as a/b where a, b have no common factors. Then $a^3 = 3b^3$. Using the previous part, we have that both a and b must be divisible by 3 which is a contradiction.
- 3. True : By the pigeonhole principle: Pigeons: 20 people. Pigeonholes: 19 possible numbers of friends. Note that there can't be people with 0 and 19 friends in the group simultaneously.

Correct Answers:

- True
- True
- True

Problem 6. 6. (8 points) local/11/100/sets/inclusion-exclusion.pg

Submit only final answers to gradescope

In a survey to 151 coffee drinkers, it was found that 74 add sugar, 65 add cream, 60 add milk, 22 add sugar and cream, 18 add sugar and milk, 7 drink coffee with the three additions and 8 prefer their coffee black (without any adddition).

Based on this information, answer the following questions. *Enter final answers only, do not type in any operations*.

- How many people add **only milk** to their coffee? ___
- How many people add cream and milk? ___
- How many people add sugar or cream, but not milk? ___
- How many people drink coffee with **only** one of the three additions? ___

Solution: *(Instructor solution preview: show the student solution after due date.)*

- How many people add **only milk** to their coffee? Ans: 26.
- How many people add cream and milk? Ans: 23
- How many people add sugar or cream, but not milk? Ans: 83.

- How many people drink coffee with **only** one of the three additions? Ans: 94.

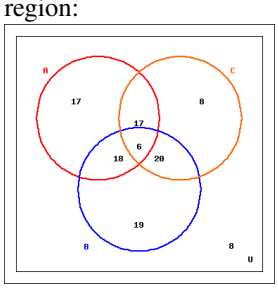
Correct Answers:

- 26
- 23
- 83
- 94

Problem 7. 7. (6 points) local/11/100/sets/vennd.pg.

Submit only final answers to gradescope

Consider the following Venn diagram that divides the universe in 8 disjoint regions. The numbers indicate the elements on each region:



How many elements are contained in each of the following sets?
A : ____
B : ____
C : ____
U : ____
$A \cap B \cap C$: ____
$\overline{A} \cup \overline{B} \cup C$: ____

Correct Answers:

- 58
- 63
- 51
- 113
- 6
- 18

Problem 8.

Submit all your work to Gradescope. No submission here

Describe an algorithm in reasonable **pseudo-code**, and using only basic arithmetic operations (sum, difference, product, division, exponents and modulo) that performs the following task:

Given two positive integers n, k, it returns the value of the binomial coefficient $\binom{n}{k}$

Note: Your algorithm must check for invalid inputs if an user enters them.

Solution: *(Instructor solution preview: show the student solution after due date.)*

Procedure: Reverse($b_1, b_2, ..., b_n$)
for i = 1,...,n
$c_i := b_{n-i+1}$
return $(c_1, \ldots, c_n)$

Problem 9.9. (6 points) local/11/100/functions/comp-formula.pg

Submit only final answers to gradescope

Given the functions $f(x) = x - 3$ and

$$g(x) = \begin{cases} 3x & \text{if } x < 2 \\ 5 & \text{if } x \geq 2 \end{cases}$$

determine the following values. If a value does not exist, write "undefined" in the blank provided.
$(f \circ g)(2) =$ ____
$(g \circ f)(3) =$ ____

Correct Answers:

- 2
- 0

Problem 10. 10. (6 points) local/11/100/functions/bigomatch-f.pg

Submit only final answers to gradescope

For each of the following functions $f(n)$, choose the answer from the answer list below that gives the **sharpest** big-O estimate for the function.

Here sharpest means that your answer is not only an upper bound but is the choice closest possible to the given function f(n) for that part.

Enter JUST THE LETTER of your chosen answer in the answer box.

Answer List:

(A) The function is $O(1)$.
(B) The function is $O(n)$.
(C) The function is $O(n^2)$
(D) The function is $O(n^3)$.
(E) The function is $O(\log(n))$.
(F) The function is $O(n\log(n))$.
(G) The function is $O(2^n)$.
(H) The function is $O(n^n)$.

1. Let $f(n)$ be the number of rounds needed (final, semifinal, quarter-finals, etc) to find the champion of a tennis tournament with n players if players are matched on a 1v1-single elimination game

____

2. $f(n) = \log(\log(n!))$

____

3. Let $f(n)$ be the number of games played on a league with n teams (each time faces each other exaclty once)

___

Solution: *(Instructor solution preview: show the student solution after due date.)*
The answers are:

1) E

2) E

3) C

Correct Answers:

- E
- E
- C

Problem 11. 11. (8 points) local/11/100/functions/one-to-one-onto.pg

Submit only final answers to gradescope

For each of the given functions $f : A \to B$, determine if it is one-to-one, onto, bijective or neither. **Check ALL that apply** .
(1) $f : \mathbb{Z} \to \mathbb{Z}.f(x) = x^3 - x.$

- A. one-to-one
- B. onto
- C. bijective
- D. neither

(2) $f : \mathbb{R} \to \mathbb{Z}.f(x) = \lfloor x \rfloor + 1.$

- A. one-to-one
- B. onto
- C. bijective
- D. neither

Solution: *(Instructor solution preview: show the student solution after due date.)*
The correct answers are:
1) D
2) B

Correct Answers:

- D
- B

Problem 12. 12. (12 points) local/11/100/modular-arithmetic/bases.pg

Final Exam - Part B

Submit justifications to gradescope to get full credit

1) Consider the decimal number 75. Enter the following representations of the given number (do NOT enter any operations in these answer boxes)

- **Binary =** ___
- **Octal =** ___
- **Hexadecimal =** ___

2) Consider the hexadecimal number $BC8$. Enter the following representations of the given number (you can use basic arithmetic operations in the boxes when entering answers if needed)

- **Binary =** ___
- **Octal =** ___
- **Decimal =** ___

Solution: *(Instructor solution preview: show the student solution after due date.)*

- 1001011
- 113
- 4B

- 101111001000
- 5710
- 3016

Correct Answers:

- 1001011
- 113
- 4B
- 1.01111E+11
- 5710
- 3016

Problem 13. 13. (10 points) local/11/100/modular-arithmetic/crt.pg

Submit justifications to gradescope to get full credit

Find the smallest positive integer solution to the following system of congruences:

$x \equiv 1 \pmod 4$

$x \equiv 4 \pmod 5$

$x \equiv 4 \pmod 7$

$x =$ _____
Solution: *(Instructor solution preview: show the student solution after due date.)*
Using the chinese remainder theorem we find that $x = 109$

Correct Answers:

- 109

Problem 14. 14. **(10 points)** local/11/100/modular-arithmetic/equation1.pg

Submit justifications to gradescope to get full credit

1) Find **all** fundamental solutions x modulo 36 (so in the set $\{0,1,2,...,35\}$) such that

$$2x+5 \equiv 9 \mod 36$$

Solutions $x =$ ___

If several solutions, enter them in the box separated by commas. If no solutions, type none.

2) Find **all** fundamental solutions x modulo 49 (so in the set $\{0,1,2,...,48\}$) such that

$$x^2 - 12x + 41 \equiv 5 \mod 49$$

Solutions $x =$ ___

If several solutions, enter them in the box separated by commas. If no solutions, type none.

Solution: *(Instructor solution preview: show the student solution after due date.)*

1. x = 2, 20
2. x = 6, 13, 20, 27, 34, 41, 48

Correct Answers:

- 2, 20
- 6, 13, 20, 27, 34, 41, 48

Problem 15. 15. **(10 points)** local/11/100/modular-arithmetic/inverse.pg

Submit justifications to gradescope to get full credit

Use the Euclidean algorithm to compute

$$\gcd(280,687) = \_\_\_\_$$

Determine the integers s and t given by the Euclidean algorithm so that

$$280s + 687t = \gcd(280,687)$$

$s =$ ___ $t =$ ___

Use the information above to find the minimum positive integer x such that

$$280x = 76 \mod 687$$

$x =$ ___

Solution: *(Instructor solution preview: show the student solution after due date.)*

Using the Euclidean algorithm and the backwards algorithm we find that

- $\gcd(280,687) = 1$.
- s = 238, t = -97.
- x = 226

Correct Answers:

- 1
- 238
- -97
- 226

Problem 16. 16. **(8 points)** local/Library/UMass-Amherst/Abstract-Algebra/PS-Congruences/Congruences10-2.pg

Submit justifications to gradescope to get full credit

What is the remainder of 5^{7513} when divided by 11?

_____

Note: You should be able to do this problem without using a calculator or computer!

Solution: *(Instructor solution preview: show the student solution after due date.)*

Using Fermat's Little Theorem $5^{10} \equiv 1 \mod 11$ find that the answer is 4.

Correct Answers:

- 4

Problem 17. 17. **(8 points)** local/11/100/modular-arithmetic/crypto/decrypt.pg

Submit justifications to gradescope to get full credit

Decrypt the message *PNWLY* which was encrypted using the cipher:

$$f(p) = (7p+9) \bmod 26$$

Message: _____

Alphabet

A=0, B=1, C=2, D=3, E=4, F=5, G=6, H=7, I =8, J=9, K=10, L=11, M=12, N=13, O=14, P=15, Q=16, R=17, S=18, T=19, U=20, V=21, W=22, X=23, Y=24, Z=25.

Correct Answers:

- MINER

Problem 18. 18. (8 points) local/11/100/combinatorics/handshake.pg

Submit justifications to gradescope to get full credit

Consider a group of 227 people. If everyone shakes hands with everyone else, how many handshakes take place?

Answer: ___

You don't need to simplify your answers, you can use the following operations in the answer box: sum, difference, product, division, exponents and factorials.

Solution: *(Instructor solution preview: show the student solution after due date.)*

There are $1+2+\cdots+226 = \dfrac{227(226)}{2} = 25651$ handshakes.

Correct Answers:

- 25651

Problem 19. 19. (9 points) local/11/100/combinatorics/tasks.pg

Submit justifications to gradescope to get full credit

16 tasks are being assigned to 5 employees.

- How many ways are there to distribute the tasks if all that matters is how many tasks are given to each employee?

Answer: ___

- How many ways are there to distribute the tasks assuming that all of them are different?

Answer: ___

- Suppose that employee A will be assigned 8 tasks, employee B will be assigned 5 tasks and employee C will be assigned 3 tasks, and again, all tasks are different. How many ways are there to distribute the tasks?

Answer: ___

You don't need to simplify your answers, you can use the following operations in the answer box: sum, difference, product, division, exponents and factorials.

Solution: *(Instructor solution preview: show the student solution after due date.)*

- $C(5+16-1,5-1) = \dfrac{20!}{(4)!(16)!} = 4845.$
- $5^{16} = 152587890625.$
- $\dfrac{16!}{8!\cdot 5!\cdot 3!} = 720720.$

Correct Answers:

- 4845
- 152587890625
- 720720

Problem 20. 20. (12 points) local/11/100/combinatorics/combinations.pg

Submit justifications to gradescope to get full credit

20. (12 points) local/11/100/combinatorics/combinations.pg

A council of 6 members is to be formed from a group of 8 Republican candidates and 7 Democrat candidates. Answer the following questions based on the information given:

- (A) How many different councils can be formed (no restriction on the council)?

Answer: ___

The council must consists of 3 Republican and 3 Democrats (**assume this fact for the remaining of the problem**).

- (B) How many different councils can be formed with this condition?

Answer: ___

- (C) Assume that there are two Democrat candidates (Mr. R and Ms. S) that refuse to serve together. How many different councils can be formed now? (Note that Mr.R or Ms. S could be part of the council).

Answer: ___

The council requires a President and a Vice-president.

- (D) How many councils can be formed if both positions can't be assigned to the same party members? (So a Republican President and Republican VP is not allowed for example).

Answer: ___

You don't need to simplify your answers, you can use the following operations in the answer box: sum, difference, product, division, exponents and factorials.

Solution: *(Instructor solution preview: show the student solution after due date.)*

- $5005 = \dfrac{15!}{6!\cdot 9!}.$
- $1960 = \dfrac{8!}{3!\cdot 5!}\cdot\dfrac{7!}{3!\cdot 4!}.$
- $280 = \dfrac{8!}{3!\cdot 5!}\cdot\dfrac{5!}{1!\cdot 4!}.$

- $35280 = 2 \cdot 8 \cdot 7 \frac{7!}{2! \cdot 5!} \cdot \frac{6!}{2! * 4!}$.

Correct Answers:

- 5005
- 1960
- 1680
- 35280

Problem 21. 21. (9 points) local/11/100/combinatorics/binomial.pg

Submit justifications to gradescope to get full credit

The **constant** term of the expansion of the expression $\left(4x^2 - \frac{2}{x^4}\right)^{24}$ is of the form

$C \cdot 4^a \cdot 2^b$

where

C = ___ a = ___ b = ___

Note:*You must enter a negative sign if any of the answers is negative.*

Solution: *(Instructor solution preview: show the student solution after due date.)*

- C = (1)C(24,16) = (1)735471
- a = 16
- b = 8

Correct Answers:

- 735471
- 16
- 8

Problem 22.

Submit all your work to Gradescope.

A certain creature has in average 3 offspring during the year. The enviroment where this creature lives is very hostile, and about 62% of the individuals die every year. Also, this creature spends one year to reach to reproductive age.

If a population of 1400 newborn individuals of this species is observed in their native environment:

1) Determine the following values (round down your answers to an integer)

- $a_0 =$___.
- $a_1 =$___.
- $a_2 =$___.
- $a_3 =$___.

2) Write a **recursive** formula a_n for the number of creatures alive n years after the observation date (do not forget the initial condition(s)). **Submit your answers to gradescope**.

Solution: *(Instructor solution preview: show the student solution after due date.)*

- 1400
- 532
- 1798
- 1289

- $a_n = \frac{(100-62)}{100}(a_{n-1} + 3a_{n-2})$
  $a_0 = 1400, a_1 = 532.$

Correct Answers:

- 1400
- 532
- 1798
- 1289

Problem 23.23. (12 points) local/11/100/induction-recursion/linear-rec.pg

Submit justifications to gradescope to get full credit

Consider the linear recursion

$$a_{n+1} = (12)a_n + (-36)a_{n-1}$$

with initial conditions $a_0 = 1, a_1 = 12$. Answer the following questions.

- (10%) Determine the value of a_3
  a_3 = ___
- (10%) The recursion is homogeneous
  - True
  - False
- (10%) Determine the degree of the recursion
  degree = ___
- (70%) The solution to the recursion is the function
  $S(n) =$___

Solution: *(Instructor solution preview: show the student solution after due date.)*

- a3 = 864
- True: It is homogeneous
- Degree 2
- $1 \cdot 6^n + 1n \cdot 6^n$

Correct Answers:

- 864
- True
- 2
- 1*6^n+1*n*6^n

Problem 24. 24. (10 points) local/11/100/induction-recursion/induction.pg

Submit all your work to Gradescope. No submission here

- (A) Use mathematical induction to prove that for any integer $n \geq 2$:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

- (B) Recall that the *Fibonacci sequence* is defined as $f_0 = 1, f_1 = 1, f_{n+1} = f_n + f_{n-1}$. Use mathematical induction to show that

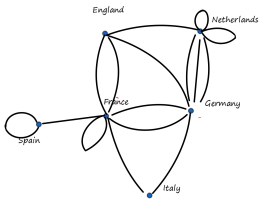
$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

In this problem, you must establish all the steps required to perform induction.

Problem 25. 25. (12 points) local/11/100/graphs/graph1.pg

Submit justifications to gradescope to get full credit

The following graph represents a transportation network between some countries in Europe. Countries are represented by vertices and routes by edges.



Label each vertex with the corresponding first letter from its country. Answer the following questions:

(A) Determine the degree of each vertex:

- deg(E) = ___
- deg(F) = ___
- deg(G) = ___
- deg(I) = ___
- deg(N) = ___
- deg(S) = ___

(B) Find the Adjacency matrix associated to the graph

—	E	F	G	I	N	S
—E	___	___	___	___	___	___
—F	___	___	___	___	___	___
—G	___	___	___	___	___	___
—I	___	___	___	___	___	___
—N	___	___	___	___	___	___
—S	___	___	___	___	___	___

(C) Determine which of the following statements is True about the network. **Submit your justifications in Gradescope**.

I. It is possible to travel from Germany across Europe and return, while covering all routes exactly once.

II. It is possible to travel from England across Europe while covering all routes exactly once.

- I is true, II is True
- I is true, II is False
- I is False, II is True
- I is False, II is False

Draw a sequence of edges showing that is possible if True, or explain clearly why it is impossible if False.

Correct Answers:

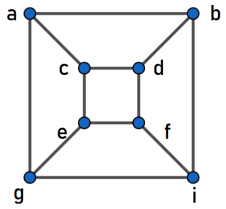
- 4
- 8
- 7
- 2
- 8
- 3
- 0
- 2
- 1
- 0
- 1
- 1
- 2
- 1
- 2
- 1
- 0
- 1
- 1

- 2
- 0
- 1
- 3
- 0
- 0
- 1
- 1
- 0
- 0
- 0
- 1
- 0
- 3
- 0
- 2
- 0
- 1
- 1
- 0
- 0
- 0
- 1
- I is False, II is True

Problem 26. 26. (8 points) local/11/100/graphs/graph2.pg

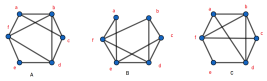
Submit justifications to gradescope to get full credit

(A) Determine if the following graph is bipartite. If it is, write a decomposition of the set of vertices. Write each group of vertices within a box separating elements by commas. If the graph is not bipartie, type 'none' in the answer boxes.



Bipartite decomposition: ___,___

(B) Consider the following graphs:



Two of these three graphs are isomorphic, determine which one is the odd man out (**submit your explanations in Gradescope**).

- Graph A
- Graph B
- Graph C

Solution: *(Instructor solution preview: show the student solution after due date.)*
1) It is bipartite with decomposition (a,d,e,i), (b,c,f,g).

2) The odd man out is graph C. It has a circuit of length 7 while de others do not.
Correct Answers:

- 1, 2, 3
- 4, 5, 6
- Graph C

Assignment FinalExam due 06/22/2023 at 12:30pm EDT

Problem 1. 1. **(3 points)** `local/1/header-f.pg`

Final Exam - Discrete Mathematics : Summer 2023

Read all the instructions below before start working on the exam

- Copy down the Honor Pledge on your paper: *I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own* and **sign** besides it.
- You have 180 minutes (3hr) or until 12:00 pm (whichever comes first) to work on your exam.
- You can change your answers anytime before submitting your exam.
- To submit your answers, click on "Grade Test" at the bottom of this page. Clicking on that button **will prompt a pasword to finish the exam immediately**. Make sure you have entered all your answers before hitting "Grade Test".
- You have **ONLY one exam attempt**. Re-attempts won't be allowed under any circumstance.
- You must show all your work to receive full-credit. Copy down on your paper both final answers and justifications **for all problems on the exam**.
- Once the exam is over, scan all your solutions in a single PDF file (including the Honor Pledge) and post the solutions on Gradescope.
- When a prompt (user and passcode) is required when clicking on "Grade Test", enter the following information: User: *ExamDone* Passcode: *HappySummer*

  - I have read, understood and agreed to all the exam instructions

Correct Answers:

- `I ... instructions`

Problem 2. 2. **(6 points)** `local/11/100/logic/contrapositive-f.pg`

FINAL EXAM - PART A

Submit only final answers to gradescope

Consider the proposition

If today is not Friday, then either people are sad or today is a not a holiday.

For each of the following propositions, determine if it is the contrapositive, converse, inverse or neither of the given proposition above.

1. *If today is Friday, then either people are sad or today is a not a holiday*

   - **A**. converse
   - **B**. inverse
   - **C**. contrapositive
   - **D**. None of the above

2. *If either people are sad or today is not a holiday, then today is not Friday*

   - **A**. converse
   - **B**. contrapositive
   - **C**. inverse
   - **D**. None of the above

Solution: *(Instructor solution preview: show the student solution after due date.)*
Given the statement $p \to q$: If today is not Friday, then either people are sad or today is a not a holiday

- The contrapositive is $\neg q \to \neg p$:If neither people are sad nor today is not a holiday, then it is Friday.
- The converse is $q \to p$: If either people are sad or today is not a holiday, then today is not Friday.
- The inverse is $\neg p \to \neg q$: If today is Friday, then neither people are sad nor today is not a holiday.

The correct solutions are then:

- Part 1: D
- Part 2: A

Correct Answers:

- D
- A

Problem 3. 3. (9 points) local/11/100/logic/predicate-tf-f.pg

Submit justifications to gradescope to get full credit
For each of the following predicates, determine if they are True or False in the universe of **Integers**. Provide your explanations on Gradescope.

1. $\forall x\ \exists y\ x \not\equiv 0 (\mod 7) \rightarrow xy \equiv 1 (\mod 7)$

- True
- False

2. $\forall x\ \forall y\ (x+y)^2 \equiv x^2 + y^2 \mod 2$

- True
- False

3. $\exists x\ \exists y\ xy \equiv 0 \mod 20 \rightarrow (x \equiv 0 \vee y \equiv 0) \mod 20$

- True
- False

Solution: *(Instructor solution preview: show the student solution after due date.)*

- True: Since 7 is prime, any non-zero number mod 7 has a multiplicative inverse mod 7
- True: $(x+y)^2 = x^2 + 2xy + y^2 \equiv x^2 + y^2 \mod 2$
- False: Take $x = 4, y = 5$

Correct Answers:

- True
- True
- False

Problem 4. 4. (8 points) local/11/100/logic/truth-tables-3.pg
Complete the following truth table to find the truth values of the formula

$$\alpha = ((p \wedge \neg q) \vee (r \rightarrow \neg q)) \rightarrow (p \vee \neg r)$$

by filling in the blanks with T (for True) or F (for False) as appropriate.

p	q	r	$(p \wedge \neg q)$	$r \rightarrow \neg q$	$(p \wedge \neg q) \vee (r \rightarrow \neg q)$	$p \vee \neg r$	α
—T	T	T	___	___	___	___	___
—T	T	F	___	___	___	___	___
—T	F	T	___	___	___	___	___
—T	F	F	___	___	___	___	___
—F	T	T	___	___	___	___	___
—F	T	F	___	___	___	___	___
—F	F	T	___	___	___	___	___
—F	F	F	___	___	___	___	___

Based on the truth table, is α a tautology?

- **A**. It is a Tautology
- **B**. It is NOT a Tautology

Solution: *(Instructor solution preview: show the student solution after due date.)*

p	q	r	$(p \vee q)$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	α
—T	T	T	F	F	F	T	T
—T	T	F	F	T	T	T	T
—T	F	T	T	T	T	T	T
—T	F	F	T	T	T	T	T
—F	T	T	F	F	F	F	T
—F	T	F	T	F	T	T	T
—F	F	T	F	T	T	F	F
—F	F	F	F	T	T	T	T

Based on the truth table, The correct answer is A

Correct Answers:

- F
- F
- F
- T
- T
- F
- T
- T
- T
- T
- T
- T

- T
- T
- T
- T
- T
- T
- T
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- T
- F
- T
- T
- F
- F
- F
- T
- T
- T
- T
- A

Problem 5. 5. (12 points) local/11/100/sets/proof-f.pg

Submit justifications to gradescope to get full credit

For each of the following statements determine if it is true or false. Provide a proof, counterexample or justification to earn full credit. Select the corresponding truth value here, **submit your justifications to gradescope and show all your steps**.

- 1. If the number n^3 is divisible by 3, then n is divisible by 3.
  - True
  - False
- 2. The number $\sqrt[3]{3}$ is irrational.
  - True
  - False
- 4. The University of Rochester has about 6000 students. There are at least 17 students that share the same birthday (month and day).
  - True
  - False

Solution: *(Instructor solution preview: show the student solution after due date.)*

- 1. True: By contrapositive, if n is not divisible by 3, there are two cases:
  $n \equiv 1 \mod 3 \rightarrow n^3 \equiv 1^3 = 1 \mod 3$.
  $n \equiv 2 \mod 3 \rightarrow n^3 \equiv 2^3 \equiv 2 \mod 3$
- 2. True: By contradiction. Assume $\sqrt[3]{3}$ is rational so write it as a/b where a, b have no common factors. Then $a^3 = 3b^3$. Using the previous part, we have that both a and b must be divisible by 3 which is a contradiction.
- 3. True : By the pigeonhole principle: Pigeons: 20 people. Pigeonholes: 19 possible numbers of friends. Note that there can't be people with 0 and 19 friends in the group simultaneously.

Correct Answers:

- True
- True
- True

Problem 6. 6. (8 points) local/11/100/sets/inclusion-exclusion.pg

Submit only final answers to gradescope

In a survey to 160 coffee drinkers, it was found that 80 add sugar, 70 add cream, 61 add milk, 25 add sugar and cream, 17 add sugar and milk, 7 drink coffee with the three additions and 8 prefer their coffee black (without any adddition).

Based on this information, answer the following questions. *Enter final answers only, do not type in any operations*.

- How many people add **only milk** to their coffee? ___
- How many people add cream and milk? ___
- How many people add sugar or cream, but not milk? ___
- How many people drink coffee with **only** one of the three additions? ___

Solution: *(Instructor solution preview: show the student solution after due date.)*

- How many people add **only milk** to their coffee? Ans: 27.
- How many people add cream and milk? Ans: 24
- How many people add sugar or cream, but not milk? Ans: 91.

- How many people drink coffee with **only** one of the three additions? Ans: 100.

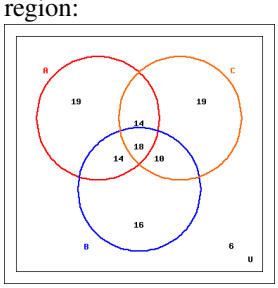
Correct Answers:

- 27
- 24
- 91
- 100

Problem 7. 7. (6 points) local/11/100/sets/vennd.pg.

Submit only final answers to gradescope

Consider the following Venn diagram that divides the universe in 8 disjoint regions. The numbers indicate the elements on each region:



How many elements are contained in each of the following sets?
A : ____
B : ____
C : ____
U : ____
$\overline{A} \cap \overline{B} \cap \overline{C}$: ____
$\overline{A} \cup \overline{C} \cap \overline{B}$: ____

Correct Answers:

- 65
- 58
- 61
- 116
- 6
- 14

Problem 8.

Submit all your work to Gradescope. No submission here

Describe an algorithm in reasonable **pseudo-code**, and using only basic arithmetic operations (sum, difference, product, division, exponents and modulo) that performs the following task:

Given a positive integer n, it returns the sum of all divisors of n

Note: Your algorithm must check for invalid inputs if an user enters them.

Solution: *(Instructor solution preview: show the student solution after due date.)*

Procedure: SUM($x_1, x_2, \ldots, x_n$)
s := 0
for i = 1,...,n
if (x_i ¿ 0) then s = s + x_i
return s

Problem 9.9. (6 points) local/11/100/functions/comp-formula.pg

Submit only final answers to gradescope

Given the functions $f(x) = x - 3$ and

$$g(x) = \begin{cases} 4x & \text{if } x < 2 \\ 5 & \text{if } x \geq 2 \end{cases}$$

determine the following values. If a value does not exist, write "undefined" in the blank provided.
$(f \circ g)(7) =$ ____
$(g \circ f)(-3) =$ ____

Correct Answers:

- 2
- -24

Problem 10. 10. (6 points) local/11/100/functions/bigomatch-f.pg

Submit only final answers to gradescope

For each of the following functions $f(n)$, choose the answer from the answer list below that gives the **sharpest** big-O estimate for the function.

Here sharpest means that your answer is not only an upper bound but is the choice closest possible to the given function f(n) for that part.

Enter JUST THE LETTER of your chosen answer in the answer box.

Answer List:

(A) The function is $O(1)$.
(B) The function is $O(n)$.
(C) The function is $O(n^2)$
(D) The function is $O(n^3)$.
(E) The function is $O(\log(n))$.
(F) The function is $O(n\log(n))$.
(G) The function is $O(2^n)$.
(H) The function is $O(n^n)$.

1. Let $f(n)$ be the number of games played on a league with n teams (each time faces each other exaclty once)

____

2. $f(n) = 5n^2 + 5n + 2\log(n^4)$

____

3. $f(n) = n\log(n^2) + 3n^2$

____

Solution: *(Instructor solution preview: show the student solution after due date.)*
The answers are:

1) C

2) C

3) C

Correct Answers:

- C
- C
- C

Problem 11. 11. (8 points) local/11/100/functions/one-to-one-onto.pg

Submit only final answers to gradescope

For each of the given functions $f : A \to B$, determine if it is one-to-one, onto, bijective or neither. **Check ALL that apply** .
(1) $f : \mathbb{N} \to \mathbb{N}. f(x) = 2^x$.

- A. one-to-one
- B. onto
- C. bijective
- D. neither

(2) $f : \mathbb{Z} \to \mathbb{Z}. f(x) = x - 3$.

- A. one-to-one
- B. onto
- C. bijective
- D. neither

Solution: *(Instructor solution preview: show the student solution after due date.)*
The correct answers are:
1) A
2) ABC

Correct Answers:

- A
- ABC

Problem 12. 12. (12 points) local/11/100/modular-arithmetic/bases.pg

Final Exam - Part B

Submit justifications to gradescope to get full credit

1) Consider the decimal number 99. Enter the following representations of the given number (do NOT enter any operations in these answer boxes)

- **Binary =** ___
- **Octal =** ___
- **Hexadecimal =** ___

2) Consider the hexadecimal number $A0F$. Enter the following representations of the given number (you can use basic arithmetic operations in the boxes when entering answers if needed)

- **Binary =** ___
- **Octal =** ___
- **Decimal =** ___

Solution: *(Instructor solution preview: show the student solution after due date.)*

- 1100011
- 143
- 63

- 101000001111
- 5017
- 2575

Correct Answers:

- 1100011
- 143
- 63
- 1.01E+11
- 5017
- 2575

Problem 13. 13. (10 points) local/11/100/modular-arithmetic/crt.pg

Submit justifications to gradescope to get full credit

Find the smallest positive integer solution to the following system of congruences:

$x \equiv 0 \pmod 2$

$x \equiv 4 \pmod 5$

$x \equiv 6 \pmod 7$

$x =$ _____

Solution: *(Instructor solution preview: show the student solution after due date.)*
Using the chinese remainder theorem we find that $x = 34$

Correct Answers:

- 34

Problem 14. 14. (10 points) local/11/100/modular-arithmetic/equation1.pg

Submit justifications to gradescope to get full credit

1) Find **all** fundamental solutions x modulo 36 (so in the set $\{0, 1, 2, ..., 35\}$) such that

$$2x + 5 \equiv 1 \mod 36$$

Solutions $x =$ ___

If several solutions, enter them in the box separated by commas. If no solutions, type none.

2) Find **all** fundamental solutions x modulo 49 (so in the set $\{0,1,2,...,48\}$) such that

$$x^2 - 12x + 44 \equiv 8 \mod 49$$

Solutions $x =$ ___

If several solutions, enter them in the box separated by commas. If no solutions, type none.

Solution: *(Instructor solution preview: show the student solution after due date.)*

1. x = 16, 34
2. x = 6, 13, 20, 27, 34, 41, 48

Correct Answers:

- 16, 34
- 6, 13, 20, 27, 34, 41, 48

Problem 15. 15. (10 points) local/11/100/modular-arithmetic/inverse.pg

Submit justifications to gradescope to get full credit

Use the Euclidean algorithm to compute

$$\gcd(107,739) = \_\_\_\_$$

Determine the integers s and t given by the Euclidean algorithm so that

$$107s + 739t = \gcd(107,739)$$

$s =$ ___ $t =$ ___

Use the information above to find the minimum positive integer x such that

$$107x = 57 \mod 739$$

$x =$ ___

Solution: *(Instructor solution preview: show the student solution after due date.)*

Using the Euclidean algorithm and the backwards algorithm we find that

- gcd(107,739) = 1.
- s = -221, t = 32.
- x = 705

Correct Answers:

- 1
- -221
- 32
- 705

Problem 16. 16. (8 points) local/Library/UMass-Amherst/Abstract-Algebra/PS-Congruences/Congruences10-2.pg

Submit justifications to gradescope to get full credit

What is the remainder of 5^{6651} when divided by 11?

___

Note: You should be able to do this problem without using a calculator or computer!

Solution: *(Instructor solution preview: show the student solution after due date.)*

Using Fermat's Little Theorem $5^{10} \equiv 1 \mod 11$ find that the answer is 5.

Correct Answers:

- 5

Problem 17. 17. (8 points) local/11/100/modular-arithmetic/crypto/decrypt.pg

Submit justifications to gradescope to get full credit

Decrypt the message *BRYJL* which was encrypted using the cipher:

$$f(p) = (11p + 19) \bmod 26$$

Message: _____

Alphabet

A=0, B=1, C=2, D=3, E=4, F=5, G=6, H=7, I =8, J=9, K=10, L=11, M=12, N=13, O=14, P=15, Q=16, R=17, S=18, T=19, U=20, V=21, W=22, X=23, Y=24, Z=25.

Correct Answers:

- WORSE

Problem 18. 18. (8 points) local/11/100/combinatorics/handshake.pg

Submit justifications to gradescope to get full credit

Consider a group of 286 people. If everyone shakes hands with everyone else, how many handshakes take place?

Answer: ___

You don't need to simplify your answers, you can use the following operations in the answer box: sum, difference, product, division, exponents and factorials.

Solution: *(Instructor solution preview: show the student solution after due date.)*

There are $1+2+\cdots+285 = \dfrac{286(285)}{2} = 40755$ handshakes.

Correct Answers:

- 40755

Problem 19. 19. (9 points) local/11/100/combinatorics/tasks.pg

Submit justifications to gradescope to get full credit

19 tasks are being assigned to 3 employees.

- How many ways are there to distribute the tasks if all that matters is how many tasks are given to each employee?

Answer: ___

- How many ways are there to distribute the tasks assuming that all of them are different?

Answer: ___

- Suppose that employee A will be assigned 9 tasks, employee B will be assigned 6 tasks and employee C will be assigned 4 tasks, and again, all tasks are different. How many ways are there to distribute the tasks?

Answer: ___

You don't need to simplify your answers, you can use the following operations in the answer box: sum, difference, product, division, exponents and factorials.

Solution: *(Instructor solution preview: show the student solution after due date.)*

- $C(3+19-1,3-1) = \dfrac{21!}{(2)!(19)!} = 210.$
- $3^{19} = 1162261467.$
- $\dfrac{19!}{9!\cdot 6!\cdot 4!} = 1.93994\times 10^7.$

Correct Answers:

- 210
- 1162261467
- 1.93994E+07

Problem 20. 20. (12 points) local/11/100/combinatorics/combinations.pg

Submit justifications to gradescope to get full credit

20. (12 points) local/11/100/combinatorics/combinations.pg

A council of 6 members is to be formed from a group of 10 Republican candidates and 9 Democrat candidates. Answer the following questions based on the information given:

- (A) How many different councils can be formed (no restriction on the council)?

Answer: ___

The council must consists of 3 Republican and 3 Democrats (**assume this fact for the remaining of the problem**).

- (B) How many different councils can be formed with this condition?

Answer: ___

- (C) Assume that there are two Democrat candidates (Mr. R and Ms. S) that refuse to serve together. How many different councils can be formed now? (Note that Mr.R or Ms. S could be part of the council).

Answer: ___

The council requires a President and a Vice-president.

- (D) How many councils can be formed if both positions can't be assigned to the same party members? (So a Republican President and Republican VP is not allowed for example).

Answer: ___

You don't need to simplify your answers, you can use the following operations in the answer box: sum, difference, product, division, exponents and factorials.

Solution: *(Instructor solution preview: show the student solution after due date.)*

- $27132 = \dfrac{19!}{6!\cdot 13!}.$
- $10080 = \dfrac{10!}{3!\cdot 7!}\cdot\dfrac{9!}{3!\cdot 6!}.$
- $840 = \dfrac{10!}{3!\cdot 7!}\cdot\dfrac{7!}{1!\cdot 6!}.$
- $181440 = 2\cdot 10\cdot 9\dfrac{9!}{2!\cdot 7!}\cdot\dfrac{8!}{2!*6!}.$

Correct Answers:

- 27132
- 10080
- 9240
- 181440

Problem 21. 21. (9 points) local/11/100/combinatorics/binomial.pg

Submit justifications to gradescope to get full credit

The **constant** term of the expansion of the expression $\left(4x^5 - \frac{3}{x^4}\right)^{27}$ is of the form

$C \cdot 4^a \cdot 3^b$

where

C = ___ a = ___ b = ___

Note:*You must enter a negative sign if any of the answers is negative.*

Solution: *(Instructor solution preview: show the student solution after due date.)*

- C = (-1)C(27,12) = (-1)17383860
- a = 12
- b = 15

Correct Answers:

- -17383860
- 12
- 15

Problem 22.

Submit all your work to Gradescope.

A certain creature has in average 4 offspring during the year. The enviroment where this creature lives is very hostile, and about 65% of the individuals die every year. Also, this creature spends one year to reach to reproductive age.

If a population of 1400 newborn individuals of this species is observed in their native environment:

1) Determine the following values (round down your answers to an integer)

- a_0 =___.
- a_1 =___.
- a_2 =___.
- a_3 =___.

2) Write a **recursive** formula a_n for the number of creatures alive n years after the observation date (do not forget the initial condition(s)). **Submit your answers to gradescope**.

Solution: *(Instructor solution preview: show the student solution after due date.)*

- 1400
- 489
- 2131
- 1430
- $a_n = \frac{(100-65)}{100}(a_{n-1} + 4a_{n-2})$
  $a_0 = 1400, a_1 = 489.$

Correct Answers:

- 1400
- 489
- 2131
- 1430

Problem 23.23. **(12 points)** local/11/100/induction-recursion/linear-rec.pg

Submit justifications to gradescope to get full credit

Consider the linear recursion

$$a_{n+1} = (6)a_n + (-5)a_{n-1}$$

with initial conditions $a_0 = -4, a_1 = -12$. Answer the following questions.

- (10%) Determine the value of a_3
  a_3 = ___
- (10%) The recursion is homogeneous
  - True
  - False
- (10%) Determine the degree of the recursion
  degree = ___
- (70%) The solution to the recursion is the function
  $S(n)$ =___

Solution: *(Instructor solution preview: show the student solution after due date.)*

- a3 = -252
- True: It is homogeneous
- Degree 2
- $-2 \cdot 1^n + (-2) \cdot 5^n$

Correct Answers:

- -252
- True
- 2
- -2*1^n+(-2)*5^n

Problem 24. 24. **(10 points)** local/11/100/induction-recursion/induction.pg

Submit all your work to Gradescope. No submission here

- (A) Use mathematical induction to prove that for any integer $n \geq 2$:

$$1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)=\frac{n(n+1)(n+2)}{3}$$

- (B) Recall that the *Fibonacci sequence* is defined as $f_0 = 1, f_1 = 1, f_{n+1} = f_n + f_{n-1}$. Use mathematical induction to show that

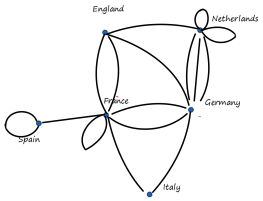
$$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$$

In this problem, you must establish all the steps required to perform induction.

Problem 25. 25. (12 points) `local/11/100/graphs/graph1.pg`

Submit justifications to gradescope to get full credit

The following graph represents a transportation network between some countries in Europe. Countries are represented by vertices and routes by edges.



Label each vertex with the corresponding first letter from its country. Answer the following questions:

(A) Determine the degree of each vertex:

- deg(E) = ___
- deg(F) = ___
- deg(G) = ___
- deg(I) = ___
- deg(N) = ___
- deg(S) = ___

(B) Find the Adjacency matrix associated to the graph

—	*E*	*F*	*G*	*I*	*N*	*S*
—*E*	___	___	___	___	___	___
—*F*	___	___	___	___	___	___
—*G*	___	___	___	___	___	___
—*I*	___	___	___	___	___	___
—*N*	___	___	___	___	___	___
—*S*	___	___	___	___	___	___

(C) Determine which of the following statements is True about the network. **Submit your justifications in Gradescope**.

I. It is possible to travel from Germany across Europe and return, while covering all routes exactly once.

II. It is possible to travel from England across Europe while covering all routes exactly once.

- I is true, II is True
- I is true, II is False
- I is False, II is True
- I is False, II is False

Draw a sequence of edges showing that is possible if True, or explain clearly why it is impossible if False.

Correct Answers:

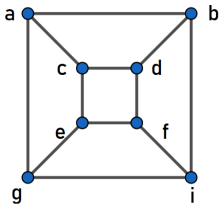
- 4
- 8
- 7
- 2
- 8
- 3
- 0
- 2
- 1
- 0
- 1
- 1
- 2
- 1
- 2
- 1
- 0
- 1
- 1
- 2
- 0
- 1
- 3
- 0
- 0
- 1
- 1
- 0
- 0
- 0
- 1
- 0

- 3
- 0
- 2
- 0
- 1
- 1
- 0
- 0
- 0
- 1
- I is False, II is True

Problem 26. 26. (8 points) local/11/100/graphs/graph2.pg

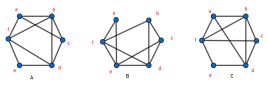
Submit justifications to gradescope to get full credit

(A) Determine if the following graph is bipartite. If it is, write a decomposition of the set of vertices. Write each group of vertices within a box separating elements by commas. If the graph is not bipartie, type 'none' in the answer boxes.



Bipartite decomposition: ___,___

(B) Consider the following graphs:



Two of these three graphs are isomorphic, determine which one is the odd man out (**submit your explanations in Gradescope**).

- Graph A
- Graph B
- Graph C

Solution: *(Instructor solution preview: show the student solution after due date.)*
1) It is bipartite with decomposition (a,d,e,i), (b,c,f,g).

2) The odd man out is graph C. It has a circuit of length 7 while de others do not.
Correct Answers:

- 1, 2, 3
- 4, 5, 6
- Graph C